Real Analysis Qualifying Review

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1. Let $1 < p < \infty$, $f \in L^p(E)$, where $E \subset \mathbb{R}^d$ is measurable.

a. Assume that for all compactly supported simple functions ϕ,

$$\int_E f(x)\phi(x)\,dx = 0.$$

Show that $f(x) = 0$ for almost every x in E.

b. Assume that for all compactly supported continuous functions $g : \mathbb{R}^d \to \mathbb{R}$

$$\int_E f(x)g(x)\,dx = 0.$$

Is $f(x) = 0$ for a.e. x in E? Justify your answer.

2. Let $f_k \in L^1(E)$, where $E \subset \mathbb{R}^d$ is measurable. Assume that

$$\sum_{k=1}^{\infty} \int_E |f_k(x)|\,dx < \infty.$$

Show that

a. $\sum_{k=1}^{\infty} f_k(x)$ is convergent for almost every x in E. Let

$$f(x) := \sum_{k=1}^{\infty} f_k(x).$$

b. Show that $f \in L^1(E)$, and

$$\sum_{k=1}^{\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx.$$

3. Assume that $f_n(x)$ is a sequence of continuous functions on $\mathbb{R}$, and

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for all } x \in \mathbb{R}.$$

Show that for any open set $O \subset \mathbb{R}$, $f^{-1}(O)$ is a F_σ set.

4. Let $f \in C^1[0,1]$.

a. Find the largest set $A \subset \mathbb{R}$, such that for each given $x \in A$,

$$\frac{f(y)}{x-y},$$

as a function of y, is Lebesgue integrable on $[0,1]$. Justify your answer.

Define
$$F(x) = \int_0^1 \frac{f(y)}{x-y}\,dy, \qquad x \in A.$$

b. Determine the largest subset of A on which F is continuous, and justify your assertion.

5. Show that there is a constant $c > 0$, such that for all real valued compactly supported C^1 functions f on $\mathbb{R}$,
$$\sup_{x\in\mathbb{R}} \int \frac{(f(x)-f(y))^4}{(x-y)^4}\,dy \le c\|f'\|^4_{L^4(\mathbb{R})}.$$
(hint: you may want to use integration by parts.)