Department of Mathematics, University of Michigan
Analysis Qualifying Exam, May 9, 2019

Morning Session, 9.00 AM-12.00

Note: $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$

Problem 1: Let $-1 < \alpha < 1$. Use residue calculus to compute

$$\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\, dx.$$

Problem 2: Let Ω be a bounded open subset of $\mathbb{C}$ and let z_1, z_2 be distinct points in Ω. Prove or disprove:

> there must be an analytic $f \colon \Omega \to \mathbb{D}$ satisfying
> $$|f(z_1) - f(z_2)| = \sup\left\{ |g(z_1) - g(z_2)| \;\middle|\; g \colon \Omega \to \mathbb{D} \text{ analytic}\right\}.$$

Problem 3: Let $f(z) = f(x+iy)$ be a complex-valued function with continuous first partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ in some domain $\Omega \subset \mathbb{C}$. Suppose that for all $\lambda \in \mathbb{C}$ the mapping $f_\lambda(z) \stackrel{\text{def}}{=} f(z) - \lambda z$ has a nonnegative Jacobian determinant (when viewed as an $\mathbb{R}^2$-valued function of two real variables). Prove that f is analytic on Ω.

Problem 4: Construct a function f analytic on a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ and satisfying

$$\int_{|z|=1} \frac{f'(z)}{f(z)}\, dz = 4\pi i$$

$$\int_{|z|=1} \frac{zf'(z)}{f(z)}\, dz = 0$$

$$\int_{|z|=1} \frac{z^2 f'(z)}{f(z)}\, dz = 5\pi i,$$

or else explain why no such function exists.

Problem 5: Let $\Omega \subset \mathbb{C}$ be a simply connected domain with $\Omega \neq \mathbb{C}$ and let $f \colon \Omega \to \Omega$ be an analytic mapping. Suppose there exist points $z_1, z_2 \in \Omega$ such that $f(z_1) = z_1$ and $f(z_2) = z_2$. Show that $f(z) = z$ for all $z \in \Omega$.

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Afternoon Session, 2.00-5.00 PM

Problem 1: Let $A \in [0,1]$ be a Lebesgue-measurable set with positive Lebesgue measure. Prove that there are two points $x, y \in A$ such that $x \neq y$ and $x - y$ is rational.

Hint: Consider $A + r$.

Problem 2: Let $f : (1,\infty) \to \mathbb{R}$ be a Lebesgue-integrable function and define $g : [0,\infty) \to \mathbb{R}$ by

$$g(y) = \int_1^\infty f(x) e^{-xy}\, dx \ , \quad y > 0 \ .$$

(a) Prove that g is a continuous function which satisfies $\lim_{y\to\infty} g(y) = 0$.
(b) Show that if f is in $L^2(1,\infty)$ then g is Lebesgue-integrable on $(0,\infty)$.

Problem 3: Consider numbers in the interval $[0,1]$ to base 8, and let E be the subset of those numbers whose decimal expansion contains the digit 5.
(a) Show that E is Lebesgue-measurable and find the Lebesgue measure of E.
(b) Define $\tau : E \to \mathbb{Z}$ by $\tau(x)$ is the first digit in the octal (base 8) expansion of x which equals 5, and define $f : E \to \mathbb{R}$ by $f(x) = 3^{\tau(x)}$. Find the values of $p > 0$ such that $f \in L^p(E)$.

Problem 4: Let $f : [0,1] \to \mathbb{R}$ be a positive function of bounded variation.
(a) Show that if $\inf f(\cdot) > 0$ then the function $g(x) = 1/f(x)$ is also of bounded variation on $[0,1]$.
(b) Give an example of a positive function $f : [0,1] \to \mathbb{R}$ of bounded variation such that $g(\cdot) = 1/f(\cdot)$ is integrable but not of bounded variation.

Problem 5: Let (X,μ) be a measure space such that $\mu(X) < \infty$. Show that

$$\lim_{n\to\infty} \int_X \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$$

if and only if $f_n \to f$ in measure as $n \to \infty$.