

Department of Mathematics, University of Michigan
Analysis Qualifying Exam, September 1, 2018
Morning Session, 9.00 AM-12.00

Problem 1:

- (a) Determine all values of λ for which the function $u(x, y)$ below is the real part of an entire function of $z = x + iy$:

$$u(x, y) = x^4 - \lambda x^3 y - 3\lambda x^2 y^2 + \lambda x y^3 + y^4.$$

- (b) In the case when the function is the real part of an analytic function, find a formula for it in terms of $z = x + iy$.

Problem 2: Suppose $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function such that $\phi(0) = 2$, $\phi(1) = 1$ and $|\phi'(z)| \leq |\phi(z)|$ for $z \in \mathbb{C}$.

- (a) Show that $\phi(z) \neq 0$ for all $z \in \mathbb{C}$.
(b) Prove that $\phi(\cdot)$ is unique and obtain an explicit formula for it.

Problem 3: Construct an explicit analytic bijection from

$$\{z \in \mathbb{C} : |z - 1|^2 < 2, |z + 1|^2 > 2\}$$

to $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. (You may write your mapping as a composition of simpler explicit mappings.)

Problem 4: Let p be a polynomial of degree $d \geq 1$ and let g be an entire function.

- (a) Find an integral representation of the function h defined by $h(w) = g(z_{1,w}) + \cdots + g(z_{d,w})$, where $z_{1,w}, \dots, z_{d,w}$ are the roots of $p(z) - w$, listed according to multiplicity.
(b) Prove that h is an entire function.

Problem 5: Let $f(z) = z^2 - 4 - e^{-3z}$ for $z \in \mathbb{C}$. Show that:

- (a) f has a unique zero x_0 on the positive real axis.
(b) x_0 is the only zero of f in the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

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Afternoon Session, 2.00-5.00 PM

Problem 1: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function which has the property that

$$(A) \quad m(|f| > \alpha) \leq \frac{1}{1 + \alpha^3} \quad \text{for } \alpha > 0.$$

- (a) Show that $|f|^p$ is integrable for $p < 3$.
(b) Give an example of a function satisfying (A) for which $|f|^3$ is not integrable.

Problem 2: Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be the periodic function with period 1 which is given in the interval $[0, 1)$ by $H(x) = 1$ if $0 \leq x < 1/2$ and $H(x) = -1$ if $1/2 \leq x < 1$. Consider the sequence of functions $H_n \in L^2(0, 1)$, $n = 1, 2, \dots$, defined by $H_n(x) = H(2^n x)$, $0 < x < 1$.

- (a) Show that

$$\lim_{n \rightarrow \infty} \int_0^1 g(x) H_n(x) dx = 0 \quad \text{for all } g \in L^2(0, 1).$$

- (b) Show that the sequence $H_n(\cdot)$, $n = 1, 2, \dots$, has no convergent subsequence in $L^2(0, 1)$.

Problem 3: Suppose that ϕ and h are \mathbb{R} -valued functions on the interval $[0, 1]$ satisfying

- ϕ is continuous on $[0, 1]$;
- $\phi(0) = 0$;
- $\phi(x) > 0$ for $x \in (0, 1]$;
- $\phi'(0)$ exists and is positive;
- h is bounded and Lebesgue-measurable on $[0, 1]$;
- h is continuous at 0.

Compute

$$\lim_{n \rightarrow \infty} n \phi'(0) \int_0^1 e^{-n\phi(x)} h(x) dm(x)$$

(with proper justification).

Hints:

- Show that $\psi(x) \stackrel{\text{def}}{=} \phi(x)/x$ has a positive lower bound.

- One approach rewrites the given integral as

$$\phi'(0) \int_{\mathbb{R}} \mathbb{1}_{[0,n]} e^{-n\phi(x/n)} h(x/n) dm(x).$$

Problem 4: Let $\phi : [0, 1] \rightarrow \mathbb{R}$ be a continuous function which has the property that

$$\limsup_{y \rightarrow x} \frac{|\phi(x) - \phi(y)|}{|x - y|^{1/2}} = \infty \quad \text{on a dense set of } x \in [0, 1].$$

Show that $\phi(\cdot)$ does not have bounded variation. (State clearly any standard lemma you quote.)

Problem 5: Consider the function $\phi : [-\pi, \pi] \rightarrow \mathbb{R}$ defined by $\phi(x) = |x|^{3/2}$ and let the Fourier series representation for $\phi(\cdot)$ be given by

$$\phi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

Show that:

- The Fourier coefficients a_n satisfy the inequality $|a_n| \leq \frac{\sqrt{\pi}}{n}$ for $n \neq 0$.
- Show further that $\lim_{n \rightarrow \infty} n a_n = 0$.