Analysis Qualifying Review

Thursday, May 3, 2018
Morning Session, 9:00 AM - Noon

N.B.: D below denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

(1) Suppose that we have

(a) simply-connected domains $\Omega_1, \Omega_2 \subset \mathbb{C}$;

(b) distinct points $z_1, w_1 \in \Omega_1$;

(c) distinct pints $z_2, w_2 \in \Omega_2$.

Show that there is an analytic map $f \colon \Omega_1 \to \Omega_2$ satisfying $f(z_1) = z_2, f(w_1) = w_2$ or an analytic map $f \colon \Omega_2 \to \Omega_1$ satisfying $f(z_2) = z_1, f(w_2) = w_1$ (or both).

(2) Let Σ be the strip $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < 1\}$, and let F be analytic on Σ, continuous on $\overline{\Sigma}$, and verifying $|F(z)| \leq 1$ on $\partial\Sigma$.

(a) Show that $|F(z)|$ is not necessarily ≤ 1 on Σ.

(b) Show that if, in addition, F verifies the hypothesis $|F(z)| \leq C\, e^{b|z|^\rho}$, for some constants $C, b > 0$ and $0 < \rho < 2$, then $|F(z)| \leq 1$ on Σ.

Hint: Consider $F_\epsilon(z) := e^{-\epsilon z^2} F(z)$, for all $\epsilon > 0$.

(3) Let f be an analytic function on D which is continuous on $\overline{D}$ with $|f(z)| \equiv 1$ on ∂D. Show that f is the restriction to D of a rational function on $\mathbb{C}$.

(4) Let $D^* := D \setminus \{0\}$ be the punctured unit disk. Let $f \colon D^* \to \mathbb{C}$ be analytic and injective.

(a) Show that $\{f(z) \colon 0 < |z| < 1/2\}$ is not dense in $\mathbb{C}$.

(b) Show that f has a meromorphic extension to D. (Do not quote Picard's theorem here.)

(5) Suppose that g, h are continuous, $\mathbb{C}$-valued and nowhere vanishing functions on $\{z \in \mathbb{C} : |z| < 2\}$, $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, *respectively.* Suppose that $f = g/h$ is analytic on the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

(a) Show that there are continuous, single-valued functions $\log g$ on $\{z \in \mathbb{C} : |z| < 2\}$, and $\log h$ on $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$.

(continued over)

(b) Show that $U = \log g - \log h$ is analytic on the annulus A.

(c) Show that f can be written as $f(z) = G(z)/H(z)$ where G, H are nowhere vanishing *analytic* functions on $\{z \in \mathbb{C} : |z| < 2\}$, $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively.

Analysis Qualifying Review

Thursday, May 3, 2018
Afternoon Session, 2:00 - 5:00 PM

N.B.: Lebesgue measure is denoted below by "m".

(1) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \geq 0$ be in $L^1(X, \mu)$. Let a set function ν be defined on $\mathcal{A}$ by $\nu(A) = \int_A f d\mu$. Show that ν is a measure on $\mathcal{A}$ and that for any ν-integrable function g,

$$\int_X g\, d\nu = \int_X g \cdot f\, d\mu.$$

(2) Provide a (detailed) proof or a (detailed) counterexample to the following statement: If E is a bounded open subset of $\mathbb{R}$ then the boundary of E has Lebesgue measure zero.

(3) Show that $\left\{f \in L^2(\mathbb{R}, m) : \int_{\mathbb{R}} |f| = \infty\right\}$ is dense in $L^2(\mathbb{R}, m)$.

(4) Let μ be a non-negative measure on the interval $(-1, 1)$ with the property that all open subintervals of $(-1, 1)$ are μ-measurable and $\mu((-1, 1)) = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous and let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n(x) = \int_{-1}^{1} f\left(x + \frac{t}{n}\right) d\mu(t)$.

(a) Show that each f_n is uniformly continuous.

(b) Show that the f_n converge uniformly to f.

(5) Let f_n be a sequence of functions in $L^\infty([0, 1], m)$ satisfying the conditions

(i) $\|f_n\|_{L^\infty([0,1],m)} \leq 1$, and

(ii) $\int_{[a,b]} f_n\, dm \to 0$ for all $0 \leq a < b \leq 1$.

(a.) Show that $\int_{[0,1]} f_n\, g\, dm \to 0$ for all $g \in L^1([0, 1], m)$.

(b.) Under assumptions (i) and (ii), does $f_n \to 0$ in $L^1([0, 1], m)$?