Analysis Qualifying Review

Saturday, January 13, 2018

Morning Session, 9:00 AM - Noon

1. (a.) Let $A \stackrel{\text{def}}{=} \{z \colon \frac{1}{2} < |z| < 1\}$. Suppose we have a sequence f_j of rational functions and a further rational function f satisfying

($i.$) none of these functions have any poles in $A \cup \{0\}$;

($ii.$) $f_j \to f$ uniformly on A.

Prove or disprove: We must have $f_j(0) \to f(0)$.

(b.) Same as problem part (a.) but with the *further* assumption that each f_j is non-zero on the closed disk $|z| \leq 1$.

2. Let u be a harmonic function on the annulus A in problem 1 above, and which is continuous on the closure $\overline{A}$. We assume $u(z) = 0$ if $|z| = 1$, and $u(z) < 0$ if $|z| = \frac{1}{2}$.

(a.) Show that in polar coordinates the Laplace operator $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, where $z = x + iy$, is expressed as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(b.) Show that $\Delta \log |z| = 0$ on $\{z \,:\, 0 < |z| < 1\}$.

(c.) Show that $u + a \log |z| \leq 0$, for a sufficiently small negative constant a.

(d.) Finally, use this last to show that $\frac{\partial u}{\partial r}(z_0) > 0$ at any point z_0 with $|z_0| = 1$.

3. Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Construct an analytic bijection $f \colon \Omega \to \Omega$ satisfying $f(1) = i$.

4. The function $f(z) = \dfrac{e^{1/z}}{z^2 - 1}$ has a Laurent expansion $\sum_{n=-\infty}^{\infty} a_n z^n$ on $0 < |z| < 1$. Find a_0.

5. Construct a bounded analytic function f on $\{z \colon 0 < |z| < 1\}$ satisfying $f\left(\frac{1}{n}\right) = \frac{n}{n+2}$ or else show that no such function exists.

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Saturday, January 13, 2018
Afternoon Session, 2:00 - 5:00 PM

N.B.: "Measure" means "Lebesgue measure" throughout.

1. Let $f_j : [0,1] \to [0,1]$ be a sequence of integrable functions satisfying $\int f_j \to 0$. Prove or disprove: we must have $f_j \to 0$ almost everywhere.

2. Prove or disprove: If $f\colon [0,1] \to \mathbb{R}$ has bounded variation and (a_j) is a decreasing sequence in $(0,1]$ with $a_j \to 0$ then $\liminf f(a_j) = \limsup f(a_j)$.

3. Let $E \subset \mathbb{R}$ be a measurable set of positive measure and let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function with positive integral. Show that there exists $t \in \mathbb{R}$ so that $\int\limits_{E+t} f > 0$.

4. Suppose we are given

(1) a bounded continuous function f on $\mathbb{R}$;

(2) an integrable function g on $\mathbb{R}$.

For $t > 0$ let $h(t) = \int\limits_{\mathbb{R}} f(tx)g(x/t)\, dm(x)$.

(a.) Must h be continuous?

(b.) Must h be bounded?

(c.) Must $\lim\limits_{t \searrow 0} h(t)$ exist?

5. Let f be a continuous function on the interval [0,1]. For $x \in [0,1)$, define

$$D^+ f(x) = \limsup_{h \searrow 0^+} \frac{f(x+h) - f(x)}{h}.$$

Show that if $D^+ f(x) \geq 0$ for all $x \in [0,1)$, then $f(1) \geq f(0)$.

Hint: Consider $f_\epsilon(x) := f(x) + \epsilon\, x$, for all real x and any $\epsilon > 0$.