Analysis Qualifying Review. May 4, 2017

Morning Session, 9:00 am - 12:00 pm

1. Let $(f_j)_{j=1}^{\infty}$ be a sequence of measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose that the series
$$\sum_{j=1}^{\infty} \mu\{x \in X \mid |f_j(x)| \geq \epsilon\}$$
converges for every $\epsilon > 0$. Prove that $f_j(x) \to 0$ almost everywhere on X.

   Solution Set $E_{j,\epsilon} = \{|f_j| \geq \epsilon\}$ and $A_\epsilon = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_{j,\epsilon}$. Then A_ϵ is measurable and $\limsup_j |f_j| \leq \epsilon$ on A_ϵ^c. Further,
$$\mu(A_\epsilon) = \lim_{n\to\infty} \mu(\bigcup_{j=n}^{\infty} E_{j,\epsilon}) \leq \lim_{n\to\infty} \sum_{j=n}^{\infty} \mu(E_{j,\epsilon}) = 0.$$
   Set $A = \bigcup_{k=1}^{\infty} A_{1/k}$. Then $\mu(A) \leq \sum_k \mu(A_k) = 0$ and $\lim_{j\to\infty} f_j(x) = 0$ for $x \in A^c$.

2. Let $E \subset [0,1]$ be the middle-third Cantor set, i.e. $E = [0,1] \setminus \bigcup_{n=1}^{\infty} U_n$, where $U_1 = (1/3, 2/3)$, $U_2 = (1/9, 2/9) \cup (7/9, 8/9)$ etc. Find a function $f \in C^\infty(\mathbb{R})$ such that $f \geq 0$ and $\{x \in \mathbb{R} \mid f(x) = 0\} = E$.

   Solution: Let $g(x)$ be the distance from a point $x \in \mathbb{R}$ to E. Then g is nonnegative with $\{g = 0\} = E$. Further, g is continuous on $\mathbb{R}$ and C^∞ on $\mathbb{R} \setminus E$. Now consider the function $\chi\colon \mathbb{R} \to \mathbb{R}$ defined by
$$\chi(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ e^{-1/t} & \text{if } t > 0 \end{cases}$$
   Then $f = \chi \circ g$ has the required properties.

3. Let $\alpha < 1$. Prove the existence of the limit
$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{1/n} e^{\alpha x}\, dx,$$
and calculate it.

   Solution: Consider the function f_n on $(0, \infty)$ defined by
$$f_n(x) = \left(1 - \frac{x}{n}\right)^n x^{1/n} e^{\alpha x} \cdot \chi_{(0,n)}$$

We have $\lim_{n\to\infty} f_n(x) = e^{-x}\cdot 1\cdot e^{\alpha x} = e^{-(1-\alpha)x}$ pointwise on $\mathbb{R}$. To estimate f_n from above, first note that $x^{1/n} \le n^{1/n} \le e^{e^{-1}}$ for $x \in (0,n)$, where the last inequality follows by checking that the maximum of the function $y^{1/y}$ on $(0,\infty)$ occurs at $y = e$. Second, we have

$$\log(1-\frac{x}{n}) \le -\frac{x}{n}$$

for $0 < x < n$. Hence

$$(1-\frac{x}{n})^n e^{\alpha x} = \exp(n\log(1-\frac{x}{n}) + \alpha x) \le e^{-(1-\alpha)x}$$

for $0 \le x < n$, so that

$$0 \le f_n(x) \le Ce^{-(1-\alpha)x}$$

for all $x \in \mathbb{R}$, where $C = e^{e^{-1}}$. Since $\int_0^\infty e^{-(1-\alpha)x}dx < \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx = \int_0^\infty e^{-(1-\alpha)x}dx = \frac{1}{1-\alpha}.$$

4. Let $\beta > 1$ and $C > 0$. Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that $|f(x)-f(y)| \le C|x-y|^\beta$ for all $x, y \in \mathbb{R}$.

   Solution: For any x, letting $y \to x$ we see that f is differentiable at x, with derivative 0. Thus $f' \equiv 0$, so that f is constant. Conversely, any constant function f clearly satisfies the condition.

5. Construct a function $f \in L^1(\mathbb{R}^n)$ such that $f \notin L^p(U)$ for any open subset $U \subset \mathbb{R}^n$ and any $p > 1$.

   Solution: Pick a dense sequence $(x_k)_{k=1}^\infty$ in $\mathbb{R}^n$. For each k, define a function f_k on $\mathbb{R}^n$ by

$$f_k(x) = \begin{cases} |x|^{-\frac{nk}{k+1}} & \text{if } |x| < 1 \\ 0 & \text{otherwise.}\end{cases}$$

   Using polar coordinates we see that

$$\int_{\mathbb{R}^n} f_k(x)dx = c_n' \int_0^1 r^{n-1-\frac{nk}{k+1}}dr = c_n(k+1),$$

   where the constants c_n' and c_n only depend on the dimension n. A similar computation also shows that f_k^p is not locally integrable at the origin for $p \ge 1+\frac{1}{k}$. Now set

$$f(x) = \sum_{k=1}^\infty 2^{-k} f_k(x-x_k).$$

Then

$$\int_{\mathbb{R}^n} f(x)dx = c_n \sum_{k=1}^{\infty} (k+1)2^{-k} < \infty.$$

On the other hand, if $p > 1$ and $U \subset \mathbb{R}^n$ is open, then $x_k \in U$ for infinitely many k, so there exists k with $x_k \in U$ and $p \geq 1 + \frac{1}{k}$. It then follows that $f \notin L^p(U)$.

Analysis Qualifying Review. May 4, 2017

Afternoon Session, 2:00 pm - 5:00 pm

1. Let $f(z)$ be an entire function such that $f(0) = 1 + \pi i$ and $\operatorname{Re} f(z) \geq 1$ when $|z| < 1$. Compute $f'(0)$.

   Solution: The origin is a local maximum of e^{-f}. It follows from the maximum modulus principle that e^{-f}, and hence also f is constant, so $f'(0) = 0$.

2. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc and $a \in \mathbb{D} \setminus \{0\}$ a point. Find all analytic functions $f(z)$ on $\mathbb{D}$ such that

   - $|f(z)| < 1$ for all $z \in \mathbb{D}$;
   - $f(a) = 0$ and $f(0) = a$.

   Solution: Recall the Schwarz Lemma: if $g\colon \mathbb{D} \to \mathbb{D}$ is analytic and $g(0) = 0$, then $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Further, if $|g(a)| = |a|$ for some $a \neq 0$, then $g(z) = \lambda z$, where $|\lambda| = 1$.

   Set $g(z) = f(\frac{a-z}{1+\bar{a}z})$. Then $g\colon \mathbb{D} \to \mathbb{D}$ is analytic, $g(0) = 0$, and $g(a) = a$. The Schwartz Lemma gives $g(z) = \lambda z$. Here $\lambda = 1$ since $g(a) = a$. Thus $g(z) = z$, i.e. $f(z) = \frac{a-z}{1-\bar{a}z}$.

3. Use residues to compute the integral $\int_0^\infty \frac{sintx}{x} dx$ for any $t \in \mathbb{R}$. Show all your steps.

   Solution: Set $J(t) = \int_0^\infty \frac{sintx}{x} dx$. Clearly $J(0) = 0$ and $J(-t) = -J(t)$, so we may assume $t > 0$. In this case, the change of variables $x \to tx$ shows that the integral is independent of t, so we may assume $t = 1$. Now compute the integral $I = \int_\gamma \frac{e^{iz}}{z} dz$, where γ consists of the following parts: $\gamma_1 := \{|z| = \epsilon, \mathrm{Im} z \geq 0\}$; $\gamma_2 := [\epsilon, R]$, $\gamma_3 := [R, R + iR]$; $\gamma_4 := [R + iR, R - iR]$; and $\gamma_5 := [R - iR, \epsilon]$. The integral I is zero since the integrand has no poles inside γ. The integral over γ_1 tends to $-\pi i$ as $\epsilon \to 0$. The integrals over γ_2, γ_3 and γ_4 tend to zero as $R \to \infty$. The sum of the integrals over γ_1 and γ_5 is equal to $2\int_\epsilon^R \frac{sinx}{x} dx$. Thus $J(t) = \pi/2$ for $t > 0$, $J(0) = 0$ and $J(t) = -\pi/2$ for $t < 0$.

4. Prove that for any real number $a > 1$, the equation $ze^{a-z} = 1$ has exactly one solution in the unit disc, and that this solution is real and positive.

   Solution: Set $f(z) = z - e^{z-a}$. When $|z| = 1$ we have $|e^{z-a}| = e^{\operatorname{Re} z - a} < 1 = |z|$, so by Rouché's theorem, f has the same number of zeros as the function z in the unit disc, namely one. Now f is real-valued on the real interval $[0, 1]$, with $f(0) = -e^{-a} < 0$ and $f(1) = 1 - e^{1-a} > 0$, so, by continuity, f has a zero on the interval $(0, 1)$.

5. Let $f(z)$ be a complex-valued C^∞ function defined on a connected open subset Ω of the complex plane. Assume that $f(z)$ and $f^2(z)$ are both harmonic (i.e. the real and imaginary parts of these functions are harmonic). Prove that either $f(z)$ or $\overline{f(z)}$ is analytic in Ω.

   Solution: A direct computation shows that $\Delta f^2 = 2f\Delta f + 2(f_x^2 + f_y^2)$, so the assumption $\Delta f = \Delta f^2 = 0$ gives $0 = f_x^2 + f_y^2 = (f_x + if_y)(f_x - if_y)$ in Ω. If $f_x - if_y \equiv 0$ in Ω, then $\bar{f}$ is analytic in Ω. On the other hand, if $f_x - if_y \not\equiv 0$, then there exists an open subset $D \subset \Omega$ where $f_x + if_y \neq 0$, and hence $f_x - if_y = 0$ on D. Thus f is analytic on D. We claim that f is in fact analytic on all of Ω. To see this, write $f = u + iv$. Then u is harmonic on Ω, and hence admits a harmonic conjugate v' on Ω, that is, $u + iv'$ is analytic on Ω. Now v' is unique up to a constant (since Ω is connected) so we may assume $v' = v$ on D. Then $v' - v$ is a real-valued harmonic function on Ω that vanishes on D, and hence must vanish everywhere. Thus $f = u + iv = u + iv'$ is analytic on Ω.