Algebra I QR Jan 2024

Problem 1. Let V be a d-dimensional vector space over $\mathbb{C}$. Let $W = \bigwedge^{d-1} V$. Show that every vector $w \in W$ is of the form $w = v_1 \wedge v_2 \wedge \cdots \wedge v_{d-1}$, where $v_i \in V$.

Problem 2. Let $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the group homomorphism given by left multiplication by the matrix

$$\begin{bmatrix} 15 & -27 & 0 \\ -9 & 45 & 15 \\ -9 & 33 & 9 \end{bmatrix}.$$

Describe the cokernel of the map f as a sum of cyclic groups.

Problem 3. Consider the three rings $R_i := \mathbb{C}[x, y]/(x^2 - y^i)$ for $i = 1, 2, 3$. Show that these three rings are pairwise non-isomorphic.

Problem 4. Suppose that X and Y are skew-symmetric $n \times n$ matrices with entries in $\mathbb{R}$. For $A, B \in \mathrm{Mat}_{n,n}(\mathbb{R})$, define $\langle A, B\rangle = \mathrm{Tr}(A^t XBY)$ where Tr denotes the trace and A^t is the transpose of A.

(1) Show that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form.

(2) If $n = 2$ and $X = Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, what is the signature of $\langle \cdot, \cdot \rangle$?

Problem 5. Let A be an integral domain and M be an A-module. We say that M is torsion-free if for $a \in A$ and $m \in M$, we have $a \cdot m = 0$ only if $a = 0$ or $m = 0$.

(a) Let A be a principal ideal domain. Suppose that M and N are torsion-free, finitely-generated A-modules. Prove that $M \otimes_A N$ is torsion-free.

(b) Let A be the ring $\mathbb{C}[x, y]$ and let M be the ideal $(x, y) \subset A$ be viewed as an A-module. Show that $M \otimes_A M$ is not torsion-free.