Algebra I QR January 2024

Problem 1. Let V be a d-dimensional vector space over $\mathbb{C}$. Let $W = \bigwedge^{d-1} V$. Show that every vector $w \in W$ is of the form $w = v_1 \wedge v_2 \wedge \cdots \wedge v_{d-1}$, where $v_i \in V$.

Solution. Let $e_1, e_2, \ldots, e_d$ be a basis of V. Then $w_i := e_1 \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_d$ for $i = 1, 2, \ldots, d$ form a basis of W. Let $w \in W$. If w is the 0-vector then clearly $w = 0 \wedge 0 \wedge \cdots \wedge 0$. Otherwise, we may write $w = \sum_{i=1}^{d} a_i w_i$ where the $a_i \in \mathbb{C}$ are not all equal to 0. By relabeling the basis $e_1, \ldots, e_d$, we may assume that $a_d \neq 0$ and by replacing w with a nonzero scalar multiple we may assume that $a_1 = 1$. Thus

$$w = a_1 w_1 + a_2 w_2 + \cdots + a_{d-1} w_{d-1} + w_d$$

for some $a_1, a_2, \ldots, a_{d-1} \in \mathbb{C}$. We claim that

$$w = v_1 \wedge v_2 \wedge \cdots \wedge v_{d-1}$$

where $v_i = e_i + (-1)^{d-i-1} a_i e_d$. This can be checked directly.

Problem 2. Let $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the group homomorphism given by left multiplication by the matrix

$$\begin{bmatrix} 15 & -27 & 0 \\ -9 & 45 & 15 \\ -9 & 33 & 9 \end{bmatrix}.$$

Describe the cokernel of the map f as a sum of cyclic groups.

Solution. By a sequence of (determinant one) row and column operations, we find that

$$\begin{bmatrix} -7 & 18 & -30 \\ 3 & -7 & 12 \\ 18 & -43 & 73 \end{bmatrix} \begin{bmatrix} 15 & -27 & 0 \\ -9 & 45 & 15 \\ -9 & 33 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

It follows that the cokernel is the direct sum $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$. (The above decomposition is not unique, and it is only necessary to compute the diagonal matrix to solve this problem.)

Problem 3. Consider the three rings $R_i := \mathbb{C}[x, y]/(x^2 - y^i)$ for $i = 1, 2, 3$. Show that these three rings are pairwise non-isomorphic.

Solution. We have $R_1 = \mathbb{C}[x, y]/(x^2 - y) \cong \mathbb{C}[y]$ which is a principal ideal domain. We have $R_2 = \mathbb{C}[x, y]/(x^2 - y^2) = \mathbb{C}[x, y]/(x + y)(x - y)$. Since $(x + y)(x - y) = 0$ in R_2 and $(x + y)$ and $(x - y)$ are both nonzero, we have that R_2 is not an integral domain, and in particular not isomorphic to R_1.

We have $R_3 = \mathbb{C}[x, y]/(x^2 - y^3)$. We claim that R_3 is an integral domain that is not a unique factorization domain. Since every principal ideal domain is a unique factorization domain, it follows that R_3 is not isomorphic to either R_1 or R_2. We now prove the claim.

The ring $\mathbb{C}[x,y]$ is a unique factorization domain. We shall show that the element $x^2 - y^3 \in \mathbb{C}[x,y]$ is irreducible. Consider any factorization $x^2 - y^3 = fg$ in $\mathbb{C}[x,y]$ where f, g are not units. Thus $f = f(x,y), g = g(x,y)$ are non-constant polynomials. By a direct calculation, we see that f, g must both involve the variable x, and by considering the coefficient of the highest power of x, we see that $f(x,y) = ax + f_1(y)$ and $g(x,y) = bx + g_1(y)$ where $a, b \in \mathbb{C}$. Thus we have the equality $x^2 - y^3 = abx^2 + x(ag_1(y) + bf_1(y)) + f_1(y)g_1(y)$. Since $f_1(y)g_1(y) = y^3$, both f_1 and g_1 are scalar multiples of powers of y. It follows that $ag_1(y) + bf_1(y)$ cannot be 0 and thus $x^2 - y^3$ is irreducible.

It follows that the ideal $(x^2 - y^3)$ is prime and R_3 is an integral domain. The ring R_3 has a grading where $\deg(x) = 3$ and $\deg(y) = 2$. The ring has no elements of degree 1, and it follows that x and y are irreducible elements of R_3. The element $x^2 \in R_3$ has the two distinct irreducible factorizations $x^2 = (x)(x) = (y)(y)(y)$. This shows that R_3 is not a unique factorization domain.

(An alternative way to show that R_3 is not a principal ideal domain is to show directly that the ideal (x, y) is not principal.)

Problem 4. Suppose that X and Y are skew-symmetric $n \times n$ matrices with entries in $\mathbb{R}$. For $A, B \in \mathrm{Mat}_{n,n}(\mathbb{R})$, define $\langle A, B\rangle = \mathrm{Tr}(A^t XBY)$ where Tr denotes the trace and A^t is the transpose of A.

(a) Show that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form.

(b) If $n = 2$ and $X = Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, what is the signature of $\langle \cdot, \cdot \rangle$?

Solution. (a) Bilinearity follows from

$$\mathrm{Tr}((\alpha A_1+\beta A_2)^t XBY) = \mathrm{Tr}((\alpha A_1)^t XBY + (\beta A_2)^t XBY) = \mathrm{Tr}((\alpha A_1)^t XBY) + \mathrm{Tr}((\beta A_2)^t XBY)$$

$$\mathrm{Tr}(A^t X(\alpha B_1+\beta B_2)Y) = \mathrm{Tr}(A^t X(\alpha B_1)Y + A^t X(\beta B_2)Y) = \mathrm{Tr}(A^t X(\alpha B_1)Y) + \mathrm{Tr}(A^t X(\beta B_2)Y)$$

and symmetry is

$$\mathrm{Tr}(A^t XBY) = \mathrm{Tr}(Y^t B^t X^t A) = \mathrm{Tr}(B^t XAY)$$

using skew-symmetry of X, Y and the fact that $\mathrm{Tr}(C) = \mathrm{Tr}(C^t)$ and $\mathrm{Tr}(CD) = \mathrm{Tr}(DC)$ for any square matrices C, D.

(b) Pick a basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathrm{Mat}_{2,2}(\mathbb{R})$. Then the matrix $M = (m_{ij} = \langle e_i, e_j \rangle)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix has eigenvalues $1, 1, -1, -1$ and thus $\langle ., . \rangle$ has signature (2,2,0) (positive, negative, zero).

Problem 5. Let A be an integral domain and M be an A-module. We say that M is torsion-free if for $a \in A$ and $m \in M$, we have $a \cdot m = 0$ only if $a = 0$ or $m = 0$.

(a) Let A be a principal ideal domain. Suppose that M and N are torsion-free, finitely-generated A-modules. Prove that $M \otimes_A N$ is torsion-free.

(b) Let A be the ring $\mathbb{C}[x, y]$ and let M be the ideal $(x, y) \subset A$ be viewed as an A-module. Show that $M \otimes_A M$ is not torsion-free.

Solution. (a) By the fundamental theorem of finitely-generated modules for principal ideal domains, M (and N) is a direct sum of modules isomorphic to either A or A/I where $I = (f)$ is a nonzero principal ideal. The latter modules are not torsion-free since $f \cdot 1 = 0$ in $A/(f)$. Thus $M \cong A^{\oplus m}$ and $N \cong A^{\oplus n}$ are free A-modules of finite rank. We compute that $M \otimes_A N \cong A^{\oplus mn}$.

(b) Let $S = M \otimes_A M$. Consider the element $x \otimes y \in S$. We have

$$x \cdot (x \otimes y) = x \otimes (xy) = (xy) \otimes x = x \cdot (y \otimes x).$$

It follows that $x \cdot (x \otimes y - y \otimes x) = 0$ in S. We claim that the element $(x \otimes y - y \otimes x) \in S$ is nonzero in S, proving that S is not torsion-free.

View S as a $\mathbb{C}$-vector space. It is the quotient of $V := M \otimes_{\mathbb{C}} M$ by the subspace W spanned by vectors of the form $(af) \otimes g - f \otimes (ag)$, where $f, g \in M$ and $a \in A$. (In fact, it suffices to take such vectors for $a = x$ or $a = y$.). Give V and W a $\mathbb{Z}$-grading by setting $\deg(x) = \deg(y) = 1$, and $\deg(f \otimes g) = \deg(f) + \deg(g)$ for homogeneous polynomials $f, g \in M$. Then V is supported in degrees $2, 3, \ldots$ while W is supported in degrees $3, 4, \ldots$. It follows that the the degree 2 component of S is isomorphic to the degree 2 component of V, which has basis $\{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}$. In particular, $x \otimes y - y \otimes x$ is nonzero in S.