Algebra 2 Exam: January 2022

Please wait until directed to begin the exam. Please use the indicated page, and its reverse side, for your solution to each problem. Extra pages are attached at the end if you need more space; please indicate if you have used them.

Please write your **identification number** here:

Have fun!

Problem 1. Let p be a prime number. Let G be a group of order p^k for $k \geq 1$ and let H be the subgroup of G generated by elements of the form g^p. Show that $H \neq G$.

Problem 2. Let K/F be a field extension of degree n. Show that there is a subgroup of $\mathrm{GL}_n(F)$ which is isomorphic to $K^\times$.

Problem 3. Let F be a field. $\mathrm{GL}_n(F)$ is the group of invertible $n \times n$ matrices with entries in F and $\mathrm{SL}_n(F)$ is the subgroup of matrices of determinant 1. Prove or disprove: There is an action of $F^\times$ on $\mathrm{SL}_n(F)$ such that $\mathrm{GL}_n(F) \cong \mathrm{SL}_n(F) \rtimes F^\times$.

Problem 4. Let $K/\mathbb{Q}$ be a Galois extension with degree 9 and at least 2 distinct subfields $\mathbb{Q} \subsetneq L_1,\, L_2 \subsetneq K$. What is $\mathrm{Gal}(K/\mathbb{Q})$?

Problem 5. Let ζ be a primitive 7-th root of unity. Give an explicit element γ of $\mathbb{Q}(\zeta)$ such that γ is not in $\mathbb{Q}$ but γ^2 is in $\mathbb{Q}$. You may assume that the cyclotomic polynomial $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ is irreducible.

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