Algebra I Exam: January 2022

Please wait until directed to begin the exam. Please use the indicated page, and its reverse side, for your solution to each problem. Extra pages are attached at the end if you need more space; please indicate if you have used them.

Please write your **identification number** here:

Have fun!

Problem 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by "rotation by 90 degrees counter-clockwise". Is T diagonalizable over $\mathbb{R}$? (Prove your answer to be correct.)

Problem 2. Fix an integer $n \geq 0$. Let $V = \mathbb{C}[x]_{\leq n} \subset \mathbb{C}[x]$ be the subspace of polynomials of degree $\leq n$. For each $\lambda \in \mathbb{C}$, consider the $\mathbb{C}$-linear operator $T_\lambda : V \to V$ determined by

$$T_\lambda(f(x)) = \frac{d}{dx}(f(x)) - \lambda f(x).$$

Calculate the rank of $T_\lambda : \mathbb{C}[x]_{\leq n} \longrightarrow \mathbb{C}[x]_{\leq n}$ as a function of λ and n.

Problem 3. Let R be a PID (principal ideal domain). Let x and y in R. Let d be a GCD of x and y (meaning that every common divisor of x and y divides d) and let m be an LCM of x and y (meaning that every common multiple of x and y is divisible by m). Show that $R/xR \oplus R/yR$ is isomorphic (as an R-module) to $R/dR \oplus R/mR$.

Problem 4. Let A be a ring, let M be an R-module and let $E = \mathrm{Hom}_A(M, M)$. Show that M can be written as a nontrivial direct sum if and only if there is an element $e \in E$, other than 0 and Id, with $e^2 = e$.

Problem 5. Let M be a 3×3 integer matrix and suppose that $\mathbb{Z}^3/M\mathbb{Z}^3 \cong \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $\bigwedge^2 M$ be the induced map $\bigwedge^2 \mathbb{Z}^3 \longrightarrow \bigwedge^2 \mathbb{Z}^3$. Compute (with proof) the abelian group $(\bigwedge^2 \mathbb{Z}^3)/(\bigwedge^2 M)(\bigwedge^2 \mathbb{Z}^3)$.

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