Algebra 2

Identification Number: ____________

Each problem occurs on a separate page. If possible, please write your solution on that page and its reverse side. More paper, and paper for scrap work, is available.

Problem 1. Let G be a finite group of order N and let X and Y be two sets on which G acts transitively. Suppose that $\mathrm{GCD}(|X|,\ |Y|) = 1$. Let G act on $X \times Y$ by $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Show that the action of G on $X \times Y$ is transitive.

Problem 2. Let G be a finite group with $|G| \equiv 2 \bmod 4$. Let s and t be two nonidentity elements of G with $s^2 = t^2 = 1$. Show that s and t are conjugate within G.

Problem 3. Let K be the field of rational functions $\mathbb{C}(x_0, x_1, x_2, x_3, x_4)$. Let F be the subfield of K consisting of functions symmetric under the permutations $(x_0, x_1, x_2, x_3 x_4) \mapsto (x_1, x_2, x_3, x_4, x_0)$ and $(x_0, x_1, x_2, x_3, x_4) \mapsto (x_0, x_4, x_3, x_2, x_1)$. How many fields L are there with $F \subseteq L \subseteq K$? (Prove your answer to be correct.)

Problem 4. Let K be a field, let $f(x)$ be a separable polynomial of degree $n \geq 3$ with coefficients in K and let L be a splitting field for $f(x)$ over K, in which $f(x)$ factors as $(x - \theta_1)(x - \theta_2) \cdots (x - \theta_n)$. Suppose that $\mathrm{Gal}(L/K)$ is the alternating group A_n. Show that θ_n lies in the field $K(\theta_1, \theta_2, \ldots, \theta_{n-2})$, but that θ_n does not lie in $K(\theta_1, \theta_2, \ldots, \theta_{n-3})$.

Problem 5. Let $p \geq 5$ be prime. We consider the following 4 subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ where, in each case, x ranges over $\mathbb{F}_p^\times$ and y ranges over $\mathbb{F}_p$:

$$G_{1,2} = \left\{\begin{bmatrix} x & y \\ 0 & x^2 \end{bmatrix}\right\} \quad G_{1,3} = \left\{\begin{bmatrix} x & y \\ 0 & x^3 \end{bmatrix}\right\} \quad G_{2,3} = \left\{\begin{bmatrix} x^2 & y \\ 0 & x^3 \end{bmatrix}\right\} \quad G_{2,1} = \left\{\begin{bmatrix} x^2 & y \\ 0 & x \end{bmatrix}\right\}$$

Which of these groups are isomorphic to each other? When you claim that groups are isomorphic, prove them to be so; when you claim that groups are not isomorphic, prove them not to be so.