ALGEBRA II

Problem 1. Let k be a positive integer. The group $\mathrm{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ consists of matrices with entries in the ring $\mathbb{Z}/2^k\mathbb{Z}$ whose determinant is a unit of $\mathbb{Z}/2^k\mathbb{Z}$. Show that $\mathrm{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ is a solvable group. You may use without proof that $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$ is solvable.

Problem 2. Let G be a group with the following presentation:

$$G = \left\langle a, b \;\middle|\; (a^2b)^5 = 1,\ a^2ba^{-1}b^{-2} \right\rangle$$

and let $[G, G]$ be the commutator subgroup of G. (The commutator subgroup is defined as the subgroup of G generated by all elements of the form $aba^{-1}b^{-1}$ for a, $b \in G$.) Compute the order of the quotient $G/[G, G]$.

Problem 3. Let L/F be a field extension and let K_1 and K_2 be two distinct subfields with $F \subset K_1, K_2 \subset L$ such that $L = K_1K_2$ and $[K_1 : F] = [K_2 : F] = 3$. Show that $[L : F]$ is either 6 or 9, and give examples to show that both values can occur.

Problem 4. Let L be the field $\mathbb{C}(x_1, x_2, x_3, x_4)$ of rational functions in four independent variables. Let $K \subset L$ be the subfield of S_4-symmetric functions. (In other words, functions in L which are symmetric in permuting the x-variables.) Give an explicit element $\theta \in L$ such that $[K(\theta) : K] = 3$.

Problem 5. Let G be a group of order $4n$ with n odd. Suppose that G contains (at least) two distinct cyclic groups of order $2n$. Show that G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/n\mathbb{Z})$.

Date: August 2022.