Algebra 2

Problem 1. Let G be a **finite** group and let $\phi : G \to G$ be a group homomorphism. For $n \geq 1$, let $\phi^n : G \to G$ denote the n-fold composition $\phi \circ \cdots \circ \phi$. Set $A = \bigcap_{n=1}^{\infty} \mathrm{Im}(\phi^n)$ and $B = \bigcup_{n=1}^{\infty} \mathrm{Ker}(\phi^n)$. Show that B is normal in G, and G is the semi-direct product of A and B.

Solution. We have $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\phi^2) \subseteq \cdots$, and each of these kernels is a normal subgroup because it is a kernel of a homomorphism, so the union of all these normal subgroups is normal.

To see that $G = A \ltimes B$, we must additionally check that $A \cap B = \{1\}$ and that $G = AB$. We first check that $A \cap B = 1$. Choose M large enough that $\mathrm{Im}(\phi^M) = \mathrm{Im}(\phi^{M+1}) = \cdots$ and $\mathrm{Ker}(\phi^M) = \mathrm{Ker}(\phi^{M+1}) = \cdots$. Let $g \in A \cap B$. Since $g \in A$, we can write $g = \phi^M(h)$ for some $h \in G$ and, since $g \in B$, we have $\phi^{2M}(h) = \phi^M(g) = 1$, showing that $h \in B$. But then, by our choice of M, we have $\phi^M(h) = 1$, so $g = 1$.

There are several ways to show that $G = AB$; here is one. Let $g \in G$ and choose M as above. Then $\phi^M(g) \in A$, so we can find some $h \in G$ with $\phi^{2M}(h) = \phi^M(g)$. Put $a = \phi^M(h)$, so $a \in A$ and we have $\phi^M(a) = \phi^M(g)$. Then $\phi^M(a^{-1}g) = 1$, so $a^{-1}g \in B$, and we see that $g = a(a^{-1}g)$ with $a \in A$ and $a^{-1}g \in B$.

Problem 2. Show that there is no simple group of order 600.

Solution. Let G be a group of order 600. The number of 5-Sylow subgroups of G must be a divisor of 24 which is 1 mod 5, and therefore must be either 1 or 6. If there is only one 5-Sylow, then this 5-Sylow is normal and hence G is not simple. Thus, we need only consider the possibility that G has six 5-Sylows. Then the action of G on the 5-Sylows gives a homomorphism $\phi : G \to S_6$. But 600 does not divide $|S_6| = 720$, so ϕ must have a kernel, and this kernel is a normal subgroup. Thus, the only way G could be simple is if ϕ is a trivial homomorphism, but then G does not act transitively on the 5-Sylows, giving a final contradiction.

Problem 3. Let K be a nontrivial extension field of $\mathbb{C}$. Show that K does not have a countable basis as a $\mathbb{C}$ vector space.

Solution. Let t be an element of K not in $\mathbf{C}$. Since $\mathbf{C}$ is algebraically closed, t generates a purely transcendental extension $\mathbf{C}(t)$. We will show that the uncountable set of elements $\{\frac{1}{t-\alpha}\}_{\alpha \in \mathbf{C}}$ is linearly independent over $\mathbf{C}$. Indeed, suppose we had a linear relation $\sum_{i=1}^{n} \beta_i \frac{1}{t-\alpha_i}$ in the field $\mathbf{C}(t)$ for some distinct α_i in $\mathbf{C}$ and some coefficients β_i in $\mathbf{C}$. Then, multiplying through by $\prod_{i=1}^{n}(t-\alpha_i)$, we see that

$$\sum_{i=1}^{n} \beta_i \prod_{j \neq i} (t - \alpha_j) = 0$$

in the ring $\mathbf{C}[t]$. Evaluating this polynomial identity at $t = \alpha_i$, we see that $\beta_i = 0$. So we have shown that $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, and there are no nontrivial linear relations between the fractions $\frac{1}{t-\alpha_i}$.

Problem 4. Let p and q be distinct prime numbers and let $K/\mathbb{Q}$ be a Galois field extension of degree $p^a q^b$ with $a, b \geq 1$. Show that there are linearly disjoint proper subfields E and F of K such that K is the compositum EF.

Solution. Let P and Q be a p-Sylow subgroup and a q-Sylow subgroup of G respectively, and let E and F be their subfields. Since $P \cap Q = \{e\}$, we have $EF = K$. Since $[E : \mathbf{Q}]$ and $[F : \mathbf{Q}]$ are relatively prime, we have $E \cap F = \mathbf{Q}$.

Problem 5. Let n be a positive integer, let $K = \mathbb{Q}(x_1, x_2, \ldots, x_n)$, and let $F \subset K$ be the subfield of functions that are symmetric in $x_1, x_2, \ldots, x_n$. Set

$$\begin{aligned} p &= x_1^2 x_2 + x_2^2 x_3 + \cdots + x_{n-1}^2 x_n + x_n^2 x_1 \\ q &= x_1 x_2^2 + x_2 x_3^2 + \cdots + x_{n-1} x_n^2 + x_n x_1^2. \end{aligned}$$

Show that q belongs to $F(p)$, the subfield of K generated by p and F.

Solution. Since F is the fixed field of the action of S_n on K, the extension K/F is Galois with Galois group S_n. We claim that the subgroup of S_n stabilizing $F(p)$ is $\langle (123 \cdots n) \rangle$. Indeed, the stabilizer of p is $\langle (12 \cdots n) \rangle$, so no larger subgroup can stabilize $F(p)$, and every other element of $F(p)$ is likewise stabilized by $\langle (12 \cdots n) \rangle$.

Thus, by the Main Theorem of Galois Theory, $F(p)$ is the fixed field of $\langle (12 \cdots n) \rangle$ and, in particular, as $\langle (12 \cdots n) \rangle$ fixes q, we must have $q \in F(p)$.