Algebra 1

Problem 1. Let V be a 2-dimensional complex vector space. What is the largest value of n for which there are vectors $v_1, \ldots, v_n$ in V such that $v_1^{\otimes 3}, \ldots, v_n^{\otimes 3}$ are linearly independent? Here $v^{\otimes 3}$ denotes the element $v \otimes v \otimes v$ of $V^{\otimes 3} = V \otimes V \otimes V$.

Solution. Let e, f be a basis for V. Consider an element $v = \alpha e + \beta f$ of V. Then

$$v^{\otimes 3} = \alpha^3 eee + \alpha^2 \beta(eef + efe + fee) + \alpha\beta^2(eff + fef + ffe) + \beta^3 fff.$$

Here we have omitted tensor symbols; thus eee means $e \otimes e \otimes e$. In other words, if we define

$$g_1 = eee, \quad g_2 = eef + efe + ffe, \quad g_3 = eff + fef + ffe, \quad g_4 = fff$$

then

$$v^{\otimes 3} = \alpha^3 g_1 + \alpha^2 \beta g_2 + \alpha\beta^2 g_3 + \beta^3 g_4.$$

We thus see that $v^{\otimes 3}$ belongs to the span of $g_1, \ldots, g_4$, which is a four dimensional space (note that the g's are linearly independent since they have no basis vectors in common). This shows that $n \le 4$.

In fact, $n = 4$. To see this, let $v_1, \ldots, v_4$ be four elements of V and write $v_i = \alpha_i e + \beta_i f$. Expressing $v_i^{\otimes 3}$ in terms of the g basis, the coefficient vectors are the rows of the following matrix:

$$\begin{pmatrix} \alpha_1^3 & \alpha_1^2\beta_1 & \alpha_1\beta_1^2 & \beta_1^3 \\ \alpha_2^3 & \alpha_2^2\beta_2 & \alpha_2\beta_2^2 & \beta_2^3 \\ \alpha_3^3 & \alpha_3^2\beta_3 & \alpha_3\beta_3^2 & \beta_3^3 \\ \alpha_4^3 & \alpha_4^2\beta_4 & \alpha_4\beta_4^2 & \beta_4^3 \end{pmatrix}$$

The $v_i^{\otimes 3}$ are linearly independent if and only if the above matrix is non-singular. We thus just need to pick the α's and β's to make the determinant non-zero. This is clearly possible, since the determinant is not the zero polynomial: the coefficient of $\alpha_1^3\alpha_2^2\alpha_3$ is non-zero (it appears in only on term when we expand the determinant). To be definite, we can take

$$(\alpha_1, \beta_1) = (1, 0), \quad (\alpha_2, \beta_2) = (1, 1), \quad (\alpha_3, \beta_3) = (1, -1), \quad (\alpha_4, \beta_4) = (0, 1).$$

Remark. For any complex vector space V, the vectors $v^{\otimes d}$ belong to and span the space $\mathrm{Sym}^d(V)$, which we identify with the S_d-invariant vectors of $V^{\otimes d}$. Thus the maximal n for which there exists linearly independent vectors $v_1^{\otimes d}, \ldots, v_n^{\otimes d}$ is given by $n = \dim \mathrm{Sym}^d(V)$. Explicitly, this is $\binom{m+d-1}{d}$ where $m = \dim(V)$.

Problem 2. Let X be an $n \times n$ matrix with entries in $\mathbf{C}$. Let

$$V = \{Y \in \mathrm{Mat}_{n\times n}(\mathbf{C}) \ : \ XY = YX\},$$

which is a vector subspace of $\mathrm{Mat}_{n\times n}(\mathbf{C})$. Show that $\dim_{\mathbf{C}} V \ge n$.

Solution 1. Regard $\mathbf{C}^n$ as a $\mathbf{C}[t]$-module with t acting by X. Then V is exactly the set of $\mathbf{C}[t]$-module endomorphisms of $\mathbf{C}^n$. Thus it suffices to prove the following statement: if M is a finite dimensional $\mathbf{C}[t]$-module then $\dim \mathrm{End}_{\mathbf{C}[t]}(M) \ge \dim M$. (Throughout this solution, "dimension" means "dimension as a $\mathbf{C}$-vector space.")

Suppose that M and N are finite dimensional $\mathbf{C}[t]$-modules. Then $\mathrm{End}_{\mathbf{C}[t]}(M \oplus N)$ contains $\mathrm{End}_{\mathbf{C}[t]}(M) \oplus \mathrm{End}_{\mathbf{C}[t]}(N)$. Thus if the result is true for M and N then it is true for $M \oplus N$.

By the structure theorem, every finite dimensional $\mathbf{C}[t]$-module is a direct sum of finite dimensional cyclic $\mathbf{C}[t]$-modules. It thus suffices to prove the result for such modules. Now, if R is any commutative ring and I is an ideal then $\mathrm{End}_R(R/I) = R/I$. In particular, for $M = \mathbf{C}[t]/I$, with I a non-zero ideal, we see that $\mathrm{End}_{\mathbf{C}[t]}(M) \cong M$, and so the result holds.

Solution 2. Write $V(X)$ for the space V in the problem. Suppose that X is a block matrix

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A has size $a \times a$ and B has size $b \times b$, with $a+b=n$. Then $V(X)$ contains $V(A) \oplus V(B)$. (Here we think of elements of $V(A) \oplus V(B)$ as block matrices similar to the above.) We thus see that if $\dim V(A) \geq a$ and $\dim V(B) \geq b$ then $\dim V(X) \geq a+b=n$. Thus it suffices to prove the result separately for A and B.

Applying this observation several times and appealing to Jordan normal form, we can reduce to the case where X is a single Jordan block. In this case, the minimal polynomial of X coincides with the characteristic polynomial and has degree n; we thus see that the elements $1, X, X^2, \ldots, X^{n-1}$ are linearly independent. Since these obviously belong to $V(X)$, we see that $\dim V(X) \geq n$.

Remark. The two solutions are essentially doing the same thing, just in different languages.

Problem 3. Let $R = \mathbf{Q}[x, y]$. Show that there are only finitely many ideals of R which contain the ideal $\langle x, y \rangle \cap \langle x-1, y-1 \rangle$.

Solution. Let $I = \langle x, y \rangle$ and $J = \langle x-1, y-1 \rangle$. It is clear that $R/I \cong \mathbf{Q}$ and $R/J \cong \mathbf{Q}$. The sum $I+J$ contains the element $x-(x-1)=1$, and is therefore the unit ideal. Thus, by the Chinese remainder theorem, we have $R/(I \cap J) \cong R/I \times R/J \cong \mathbf{Q} \times \mathbf{Q}$. Now, the ideals of R containing $I \cap J$ are in bijective correspondence with ideals of $R/(I \cap J)$; by the above, these ideals are in bijective correspondence with ideals of $\mathbf{Q} \times \mathbf{Q}$. The ring $\mathbf{Q} \times \mathbf{Q}$ has exactly four ideals: the zero ideal, the unit ideal, the ideal generated by $(1, 0)$, and the ideal generated by $(0, 1)$. Thus the result follows.

Problem 4. Let A be a finite abelian group such that $a^{10} = 1$ for all a in A. Suppose that A has exactly 168 elements of order 10. What is the order of A?

Solution. We write A additively; thus we have $10x = 0$ for all $x \in A$. By the structure theorem, we have $A \cong \mathbf{Z}/p_1^{e_1} \times \cdots \times \mathbf{Z}/p_r^{e_r}$ for prime numbers $p_1, \ldots, p_r$ and positive integers $e_1, \ldots, e_r$. Since $10x = 0$ for all $x \in A$, we must have $p_i^{e_i} \mid 10$ for all i, and so $p_i \in \{2, 5\}$ and $e_i = 1$. We thus have $A \cong (\mathbf{Z}/2)^n \times (\mathbf{Z}/5)^m$ for non-negative integers n and m. Consider an element $x = (y, z)$ of A. Then x has order 10 if and only if y and z are both non-zero. We thus see that the number of elements of A of order 10 is $(2^n-1)(5^m-1)$. We therefore have $(2^n-1)(5^m-1) = 168$. The only solution to this equation is $(n, m) = (3, 2)$. (Reason: For $m \geq 3$, the number $5^m - 1$ does not divide 168, so m must be 1 or 2. Since $168/(5^1-1) = 42$ is not of the form $2^n - 1$, we cannot have $m = 1$.) Hence $A \cong (\mathbf{Z}/2)^3 \times (\mathbf{Z}/5)^2$ has order $2^3 \cdot 5^2 = 200$.

Problem 5. Let $S = \mathbf{Q}[t]$. We'll write elements of $S^{\oplus 2}$ as column vectors. Define the following S-modules:

$$\begin{array}{rcl} M_1 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right]\right) \\ M_2 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t-1 \end{smallmatrix}\right]\right) \\ M_3 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ -1 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right]\right) \\ M_4 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ -1 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t-1 \end{smallmatrix}\right]\right) \end{array}.$$

Two of these modules are isomorphic to each other. Prove that they are isomorphic, and show that the other pairs of modules are nonisomorphic.

Solution. We decompose each of the modules according to the structure theorem. Clearly, we have

$$M_1 = S/tS \oplus S/tS, \qquad M_2 = S/tS \oplus S/(t-1)S.$$

We now consider M_3. Let $e_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ be the standard basis of $S^{\oplus 2}$. Let $v_1 = te_1 - e_2$ and $v_2 = te_2$, so that M_3 is the quotient of $S^{\oplus 2}$ by the submodule generated by v_1 and v_2. Now, $\{e_1, v_1\}$ forms a basis of $S^{\oplus 2}$, and we have $v_2 = t^2e_1 - tv_1$. We thus find

$$M_3 = (Se_1 \oplus Sv_1)/(Sv_1 + Sv_2) = Se_1/(St^2e_1) \cong S/(t^2).$$

Finally, consider M_4. Let $w_1 = te_1 - e_2$ and $w_2 = (t-1)e_2$, so that M_4 is the quotient of $S^{\oplus 2}$ by the submodule generated by w_1 and w_2. As before, $\{e_1, w_1\}$ is a basis for $S^{\oplus 2}$, and we have $w_2 = (t-1)(te_1 - w_1)$. We thus find

$$M_4 = (Se_1 \oplus Sw_1)/(Sw_1 + S(t-1)(te_1 - w_1)) = Se_1/(S(t-1)te_1) \cong S/(t(t-1)).$$

We thus see that M_2 and M_4 are isomorphic (by the Chinese remainder theorem). All other pairs are non-isomorphic by the uniqueness part of the structure theorem. This can also be seen directly by considering annihilators: the annihilator of M_1 is (t), of $M_2 \cong M_4$ is $(t(t-1))$, and of M_3 is (t^2).