Algebra II Exam – May 2021

Notation: $\mathbb{C}$ and $\mathbb{R}$ denote the fields of real and complex numbers, $\mathbb{F}_p$ denotes the finite field with p elements, and S_n and A_n denote the symmetric and alternating groups.

Problem 1. Give examples of groups G_1 and G_2 of order 8 such that:

(1) G_1 is a semi-direct product of C_2 and C_4, with C_4 normal, but is not isomorphic to the direct product $C_2 \times C_4$.

(2) G_2 contains a cyclic group of order 4, but is not isomorphic to a semi-direct product of C_2 and C_4.

Here C_n denotes the cyclic group of order n. Be sure to rigorously justify your assertions.

Problem 2. Let G be a finite group of order n. Let G act on itself by left multiplication, and let $\phi\colon G \to S_n$ be the homomorphism associated to this action. Show that $\mathrm{im}(\phi) \subset A_n$ if and only if (1) n is odd; or (2) n is even and the 2-Sylow subgroups of G are *not* cyclic.

Problem 3. Let n be a positive integer. Show that $\mathbb{C}(t)/\mathbb{R}(t^n)$ is a Galois extension, and determine its Galois group. Here t is an indeterminate and $\mathbb{C}(t)$ is the rational function field.

Problem 4. Suppose that p is a Fermat prime, i.e., p has the form $2^r + 1$ for some positive integer r. Let $a, b \in \mathbb{F}_p^\times$. Show that either $a = b^n$ for some integer n, or $b = a^m$ for some integer m.

Problem 5. Let F be a field of characteristic $\neq 2$, and let a, b, and c be non-zero elements of F such that a, b, c, ab, ac, bc, and abc are all non-squares in F. Show that $F(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is a degree 8 extension of F.