Algebra I Exam – May 2021

Notation: $\mathbb{C}$ and $\mathbb{Q}$ denote the fields of complex and rational numbers.

Problem 1. Let I be the ideal of $\mathbb{C}[x, y, z]$ generated by the elements

$$x + 2y - z, \quad 2x + y + z, \quad (x + y + 3z)(1 + 2x - y + 2z).$$

Find all maximal ideals that contain I.

Solution. From the first two equations, we obtain that

$$\tfrac{1}{3}\left((x + 2y - z) + (2x + y + z)\right) = x + y$$

and

$$\tfrac{1}{3}\left(-(x + 2y - z) + 2(2x + y + z)\right) = x + z$$

are in the ideal. So, in the quotient ring, $x = -y = -z$. Thus, the quotient ring is isomorphic to $\mathbb{C}[x]/(-3x)(1 + x) = \mathbb{C}[x]/(x(1 + x))$. By the Chinese remainder theorem, this is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. The two maximal ideals correspond to the two projections onto copies of $\mathbb{C}$. In one quotient, $x = 0$, so $y = z = 0$, and the ideal is $\langle x, y, z\rangle$. In the other quotient, $x = -1$, so $y = z = 1$ and the ideal is $\langle x + 1, y - 1, z - 1\rangle$.

Problem 2. Let V be a non-zero complex vector space, let n be a positive integer, let $\alpha \in \bigwedge^n V$, and let v be a non-zero vector in V. Show that $\alpha \wedge v = 0$ if and only if $\alpha = \beta \wedge v$ for some $\beta \in \bigwedge^{n-1} V$.

Solution. Complete the vector v to a basis $v = v_1, v_2, v_3, \ldots$ of V. Express α as $\sum a_{i_1 i_2 \cdots i_n} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n}$. The vectors $v_1 \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n}$, for $1 < i_1 < i_2 < \cdots < i_n$ are linearly independent, so that only way that $v_1 \wedge \alpha$ is 0 is if v_1 appears in every summand, in which case it can be factored out of α.

Problem 3. Let R be a commutative ring containing the field $\mathbb{C}$. Suppose that

$$0 \to N \to E \to M \to 0$$

is a short exact sequence of R-modules such that N and M are non-isomorphic and one-dimensional over $\mathbb{C}$. Show that the sequence splits (as a sequence of R-modules).

Solution. Let $\mathfrak{m} = \{x \in R : xM = 0\}$ and let $\mathfrak{n} = \{x \in R : xN = 0\}$. Since M and N are one dimensional, $\mathfrak{m}$ and $\mathfrak{n}$ are maximal ideals; since $M \not\cong N$, they are distinct maximal ideals, so, by the Chinese remainder theorem, they are coprime. Therefore, there is an element $x \in R$ which acts by 0 on M and by 1 on N.

The element x has one dimensional kernel when acting on E. (One way to see this is to choose a basis for E where the image of N is the first basis vector; then every element of R must act by a matrix of the form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$. Then x must act by a matrix of the form $\left[\begin{smallmatrix} 1 & * \\ 0 & 0 \end{smallmatrix}\right]$.) Let K be the kernel of x. Because R is commutative, for any $y \in R$ and any $k \in K$, we have $x(yk) = yxk = y0 = 0$, so $yk \in K$ again. So every $y \in R$ maps K to K, and we see that K is a submodule of E, which maps isomorphically onto M. In other words, inverting the map $K \to M$ gives a splitting.

Problem 4. Let

$$0 \to N \to M \to \mathbb{Q} \to 0$$

be a short exact sequence of abelian groups. Show that the natural map $N/kN \to M/kM$ is an isomorphism for any positive integer k.

Solution. We show separately that $N/kN \to M/kM$ is injective, and that it is surjective. Let π be the map $M \to \mathbb{Q}$.

Proof of injectivity: Let $y \in N$ and suppose that $y = kx$ for some $x \in M$.We know that $\pi(y) = 0$ so $k\pi(x) = 0$. But this implies that $\pi(x) = 0$, so $x \in M$. We see that y is 0 in M/kM.

Proof of surjectivity: Let $x \in M$; we want to show that there are $y \in N$ and $z \in M$ such that $x = y + kz$. Since $\pi : M \to \mathbb{Q}$ is surjective, we can find $z \in M$ with $\pi(z) = \frac{1}{k}\pi(x)$. So $\pi(x - kz) = 0$. Setting $y = x - kz$, we deduce that $y \in M$ and we have succeeded.

Problem 5. Let R be a commutative ring and let $f_1, f_2, \ldots$ be an infinite sequence of elements in R. Suppose that for each $N \geq 1$ there exists a field K_N and a ring homomorphism $\phi_N \colon R \to K_N$ such that $\phi_N(f_1) = \cdots = \phi_N(f_N) = 0$. Show that there exists a field K and a ring homomorphism $\phi \colon R \to K$ such that $\phi(f_i) = 0$ for all $i \geq 1$.

Solution. Let I be the ideal generated by all the f_i. We first show that $I \neq (1)$. If, for the sake of contradiction, we had $1 \in I$, then there would be $g_1, g_2, \ldots, g_N$ in R such that $f_1g_1 + f_2g_2 + \cdots + f_Ng_N = 1$. But then $\phi_N(f_1g_1 + f_2g_2 + \cdots + f_Ng_N) = 0 \cdot \phi_N(g_1) + 0 \cdot \phi_N(g_2) + \cdots + 0 \cdot \phi_N(g_N) = 0$, contradicting that $\phi_N(1) = 1$.

Now that we know that $I \neq (1)$, let $\mathfrak{m}$ be a maximal ideal containing I. The map $R \to R/\mathfrak{m}$ is the required map.