Algebra II Exam – January 2021

Problem 1. Let G be a *finite* group and let $\phi : G \to G$ be a group homomorphism. For $n \geq 1$, let $\phi^n : G \to G$ denote the n-fold composition $\phi \circ \cdots \circ \phi$. Set $A = \bigcap_{n=1}^{\infty} \mathrm{Im}(\phi^n)$ and $B = \bigcup_{n=1}^{\infty} \mathrm{Ker}(\phi^n)$. Show that B is normal in G, and G is the semi-direct product of A and B.

Problem 2. Show that there is no simple group of order 600.

Problem 3. Let K be a nontrivial extension field of $\mathbb{C}$. Show that K does not have a countable basis as a $\mathbb{C}$ vector space.

Problem 4. Let p and q be distinct prime numbers and let $K/\mathbb{Q}$ be a Galois field extension of degree $p^a q^b$ with $a, b \geq 1$. Show that there are linearly disjoint proper subfields E and F of K such that K is the compositum EF.

Problem 5. Let n be a positive integer, let $K = \mathbb{Q}(x_1, x_2, \ldots, x_n)$, and let $F \subset K$ be the subfield of functions that are symmetric in x_1, x_2, $\ldots$, x_n. Set

$$\begin{aligned} p &= x_1^2 x_2 + x_2^2 x_3 + \cdots + x_{n-1}^2 x_n + x_n^2 x_1 \\ q &= x_1 x_2^2 + x_2 x_3^2 + \cdots + x_{n-1} x_n^2 + x_n x_1^2. \end{aligned}$$

Show that q belongs to $F(p)$, the subfield of K generated by p and F.