ALGEBRA I

We use the following standard notation: $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers, and $\mathbb{F}_q$ is the finite field with q elements (where $q = p^e$ for some prime p and $e \geq 1$).

(1) Let N be a positive integer with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ (where the p_i's are distinct prime numbers, and the exponents e_i are positive). How many solutions to the equation $x^2 = x$ are there in the ring $\mathbb{Z}/N\mathbb{Z}$?

(2) An element x of a ring is called **nilpotent** if there is a positive integer N with $x^N = 0$. Show that, in a commutative ring, the set of nilpotent elements form an ideal.

(3) (a) Let $A = \{(x, y, z) \in \mathbb{Z}^3 : x \equiv y \equiv z \bmod 3\}$. Give three vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ such that $A = \mathbb{Z}\vec{u}_1 \oplus \mathbb{Z}\vec{u}_2 \oplus \mathbb{Z}\vec{u}_2$.

(b) Let $B = \{(x, y, z) \in \mathbb{Z}^3 : x + y + z \equiv 0 \bmod 3\}$. Give three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ such that $B = \mathbb{Z}\vec{v}_1 \oplus \mathbb{Z}\vec{v}_2 \oplus \mathbb{Z}\vec{v}_2$.

(c) Describe the abelian group B/A explicitly as a product of one or more cyclic groups.

(4) Let V be a finite dimensional vector space over a field k. Let $T : V \to V$ be a k-linear map of rank r. Calculate the rank of $\bigwedge^n T : \bigwedge^n V \to \bigwedge^n V$ for all n.

(5) Show that $\mathbb{Z}[x]$ and $\mathbb{Z}[x, x^{-1}]$ are not isomorphic as rings.

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