Qualifying Exam Algebra May 2020
Morning

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{F}_p$ are the integers, the rationals, the real numbers, the complex numbers and the field with p elements respectively.

(1) For a nonnegative integer n and a complex number λ, let $J_n(\lambda)$ be the $n \times n$ Jordan block with eigenvalue λ. What is the Jordan canonical form of $J_n(\lambda)^2$?

(2) (a) Show that a group of order $2^n \cdot 11$ is solvable if $n < 10$.
(b) Show that any group G of order $2^n \cdot 11$ is solvable if $n \geq 10$. (Hint: Define a group homomorphism $\varphi : G \to S_{11}$ and show that the kernel and image of φ are solvable.)

(3) Let R be a commutative ring equipped with an element f. Suppose that R is an integral domain, (f) is a prime ideal, and that the localization R_f is a field. (Here R_f is the localization $S^{-1}R$ of R with respect to the multiplicative monoid $S = \{1, f, f^2, f^3, \dots\}$.) Show that (f) is a maximal ideal in R.

(4) Suppose that p is a prime number, let K be the splitting field of the polynomial $f(x) = x^5 - 2px + p$ over $\mathbb{Q}$ and let G be the Galois group of K over $\mathbb{Q}$.
(a) Show that G contains a 5-cycle.
(b) Show that G contains a 2-cycle. (You may use the fact that $f(x)$ has exactly 3 real roots.)
(c) What is the degree of the field extension $K/\mathbb{Q}$?

(5) Let $R = \mathbb{C}[x, y]/(x^3, y^3)$.
(a) What are the zero divisors in R?
(b) For every integer $n \geq 1$, describe all elements $a \in R$ with $a^n = 1$.

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Afternoon

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{F}_p$ are the integers, the rationals, the real numbers, the complex numbers and the field with p elements respectively.

(1) Suppose that V and W are $\mathbb{R}$-vector spaces of dimension n and m, and suppose that $L : V \to W$ is a linear map.
   (a) Show that there is a unique linear map $M : \bigwedge^2 V \to \bigwedge^2 W$ such that $M(v_1 \wedge v_2) = Lv_1 \wedge Lv_2$ for all $v_1, v_2 \in V$.
   (b) Let k be the dimension of the kernel of L. Express the rank of the kernel of M in terms of k, n and m.

(2) Describe the automorphism groups of the following rings:
   (a) $\mathbb{Z}[x]$;
   (b) $\mathbb{F}_9$.

(3) Consider the polynomial $p(x) = x^{10} + x^2 + 1 \in \mathbb{F}_2[x]$ and let K be its splitting field.
   (a) Write $p(x)$ as a product of irreducible polynomials. (Prove your answer.)
   (b) How many elements does K have?

(4) For a prime number p, show that all elements in the group $\mathrm{GL}_2(\mathbb{F}_p)$ of order p are in the same conjugacy class.

(5) Define

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & -3 \\ 0 & -3 & 4 \end{pmatrix}.$$

Determine whether there exists a real 3×3 matrix C with $CAC^T = B$.