QR Exam Algebra, January 7, 2020, Morning

(1) Suppose that $A, B \in \mathrm{Mat}_{n,n}(K)$ are $n \times n$ matrices with entries in a field K such that $\mathrm{Mat}_{n,n}(K)$ is spanned by all products $A^i B^j$, $i, j \geq 0$. (By convention, $A^0 = B^0 = I$, the identity.) Show that A has rank at least $n - 1$.

(2) Let p be a prime and let $\mathbb{F}_p$ be the field with p elements.
  (a) How many elements does the group $\mathrm{SL}_2(\mathbb{F}_p)$ have?
  (b) Show that $\mathrm{SL}_2(\mathbb{F}_p)$ has an element of order $p + 1$.
  (c) Show that for every odd prime number ℓ, the ℓ-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ is cyclic.

(3) Show that for every $T \in \bigwedge^3 \mathbb{C}^4$ there exist $v_1, v_2, v_3 \in \mathbb{C}^4$ with $T = v_1 \wedge v_2 \wedge v_3$.

(4) Let $f(x) \in \mathbb{Q}[x]$ be a polynomial and $\zeta \in \mathbb{C}$ be a root of unity. Show that $f(\zeta) \neq 2^{1/3}$. (Hint: You may want to use Galois Theory.)

(5) Let $R = \mathbb{Z}[x]/(x^2 + ax + b)$ where a, b are integers.
  (a) Find explicit conditions in terms of a and b for which R an integral domain?
  (b) Find explicit conditions in terms of a and b for which R isomorphic to a product of two integral domains?

QR Exam Algebra, January 7, 2020, Afternoon

(1) Suppose that $A, B \in \mathrm{Mat}_{n,n}(\mathbb{R})$ are real symmetric matrices, and A is positive definite. Show that there exists a real number λ for which $\lambda A + B$ is positive definite.

(2) How many distinct subgroups $L \subseteq \mathbb{Z}^4$ are there for which $\mathbb{Z}^4/L$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$?

(3) Suppose that $n \geq 3$ and let $V = \{f(x) \in \mathbb{R}[x] \mid \deg(f(x)) \leq n\}$ be the $\mathbb{R}$-vector space of polynomials of degree at most n.
   (a) Define a linear map $D : V \to V$ by $D(f(x)) = f'(x) = \frac{d}{dx}f(x)$. Is D diagonalizable over $\mathbb{R}$? Is D diagonalizable over $\mathbb{C}$?
   (b) Define a linear map $E : V \to V$ by $E(f(x)) = f'(x) + x^n f(0)$. Is E diagonalizable over $\mathbb{R}$? Is E diagonalizable over $\mathbb{C}$?

(4) Suppose that $K \subseteq \mathbb{C}$ is a subfield, and $\alpha, \beta \in \mathbb{C}$ are algebraic over K such that the field extensions $K(\alpha)/K$ and $K(\beta)/K$ are Galois and the degrees of these field extensions are relatively prime. Show that $K(\alpha, \beta) = K(\alpha + \beta)$.

(5) We call a group G *hybrid* if there are two nontrivial subgroups H_1 and H_2 (not necessarily normal) such that $H_1 \cap H_2 = \{e\}$ and $H_1 H_2 = G$. (Here $H_1 H_2$ is the set of products $\{h_1 h_2 \mid h_1 \in H_1, h_2 \in H_2\}$.)
   (a) Show that the symmetric group S_n is hybrid for $n \geq 3$.
   (b) Suppose that $|G| = p^k q^\ell$ where p, q are distinct primes and $k, \ell > 0$. Prove that G is hybrid.