Algebra I Exam – Fall 2020

The symbols $\mathbb{F}_p$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote the finite field with p elements, the integers, the rational numbers, and the complex numbers

Problem 1. Let V and W be finite dimensional complex vector spaces, and let $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Let T be the tensor $v_1 \otimes w_1 + v_2 \otimes w_2$ in $V \otimes W$. Show that, if T is of the form $x \otimes y$ for some $x \in V$ and $y \in W$, then either v_1 and v_2 are linearly dependent or else w_1 and w_2 are linearly dependent.

Problem 2. Let I be the ideal $\langle x^2+1, y^2+1 \rangle$ in the ring $\mathbb{Q}[x, y]$. Show that I is not prime, and give a prime ideal containing I. (We remind the reader that the ideal (1) is not considered to be prime.)

Problem 3. Let $p(x, y)$ be an irreducible polynomial with complex coefficients. Let R be the subring of $\mathbb{C}(x, y)$ consisting of all rational functions $\frac{f(x,y)}{g(x,y)}$ such that $p(x, y) \nmid g(x, y)$. Show that every ideal of R is principal.

Problem 4. Counting up to isomorphism, how many abelian groups G are there such that G is generated by 3 elements and $g^4 = 1$ for all $g \in G$?

Problem 5. Let A be a 3×3 integer matrix. Suppose that, considered as a matrix over $\mathbb{C}$, the matrix A has Jordan form

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Let p be a prime integer and let $\bar{A}$ be the reduction of A modulo p. What are the possible Jordan forms of $\bar{A}$, considered as a matrix over the algebraic closure of $\mathbb{F}_p$?