QR Algebra

September 1, 2019

Morning

(1) Suppose that A is a complex $n \times n$ matrix. Show that A is nilpotent if and only if A and $2A$ are similar.

(2) Suppose that G is a finite group of order $|G| = p^d n$ where d and n are positive integers and p is a prime that does not divide n. Show that G contains an element of order p such that the cardinality of its conjugacy class divides n.

(3) For each of the following assertions concerning abelian groups, give an example of a nonzero abelian group A satisfying this assertion.
  (a) $A \otimes_{\mathbb{Z}} A$ is isomorphic to $A \oplus A$;
  (b) A is not finitely generated and $A \otimes_{\mathbb{Z}} A$ is isomorphic to A;
  (c) $A \otimes_{\mathbb{Z}} A = 0$.

(4) Determine whether the field extension $\mathbb{Q}(\sqrt[6]{-3})/\mathbb{Q}$ is Galois. If it is Galois, determine the Galois group.

(5) Suppose that d is a positive integer and consider the ring $R = \mathbb{Z}[i]/(3^d)$. Here, $\mathbb{Z}[i]$ is the ring of Gaussian integers with $i^2 = -1$, and (3^d) is the principal ideal in $\mathbb{Z}[i]$ generated by 3^d.
  (a) How many elements does R have?
  (b) How many elements does the group of units $R^\times$ have?

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Afternoon

(1) Describe all prime ideals in $\mathbb{C}[x,y,z]/(y^4 - z^3, y^2)$.

(2) Suppose that V is an n-dimensional complex vector space and $A : V \to V$ is a linear map. Consider the linear map $B = A \otimes I - I \otimes A$ from $V \otimes V$ to itself, where I is the identity map. Show that the rank of B is at most $n^2 - n$.

(3) For each of the following statements, prove that it is true or give a counterexample together with an explanation of why it is a counterexample.
   (a) If R is a commutative ring with $1 \neq 0$ and every submodule of R (viewed as an R-module) is free, then R is a Principal Ideal Domain (PID).
   (b) Any PID must either have 1 or infinitely many prime ideals.

(4) In a finite group G we have $g^2h^2 = h^2g^2$ for all $g, h \in G$. Show that the group G is solvable.

(5) Consider the polynomial $p(x) = x^{44} - 1 \in \mathbb{F}_3[x]$.
   (a) Show that $p(x)$ splits over the field $\mathbb{F}_{3^{10}}$.
   (b) How many roots does $p(x)$ have in each of the fields $\mathbb{F}_3$, $\mathbb{F}_{3^2}$, $\mathbb{F}_{3^5}$?
   (c) If we write $p(x)$ as a product of irreducible polynomials, how many factors of degree 10 do we have? (You do not have to find an explicit factorization.)