QR Exam Algebra
May 2, 2018
Morning

Justify your answers.

(1) Classify all finite groups G (up to isomorphism) that have only one automorphism.

(2) Suppose that F is a field, $p(x) \in F[x]$ is a separable, irreducible polynomial of degree 3 with roots $\alpha_1, \alpha_2, \alpha_3$.
  (a) Show that if the characteristic of F is not 2 or 3, then $F(\alpha_1, \alpha_2, \alpha_3) = F(\alpha_1 - \alpha_2)$.
  (b) Show that if F has characteristic 3, then it is possible that $F(\alpha_1, \alpha_2, \alpha_3) \neq F(\alpha_1 - \alpha_2)$.

(3) Suppose that A is a 2×2 matrix with real entries that is conjugate to its square A^2. What are the possible rational canonical forms for A?

(4) Let R be the ring

$$\mathbb{Z}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Z}\}$$

and $I = (5)$ be the ideal of R generated by 5. Write $R/(5)$ as a product of fields.

(5) Suppose that p, q, r are distinct prime numbers, and $\Phi_{qr}(x) \in \mathbb{Z}[x]$ is the qr-th cyclotomic polynomial. For which p, q, r is $\Phi_{qr}(x)$ irreducible as a polynomial in $\mathbb{F}_p[x]$ after reducing its coefficients modulo p?

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Afternoon

Justify your answers.

(1) Let $K/\mathbb{Q}$ be a field extension, and suppose that $\alpha, \beta \in K$ satisfy $K = \mathbb{Q}(\alpha, \beta)$ and $\alpha^2 = \beta^3$.
  (a) Show that if $\beta \in \mathbb{Q}(\alpha)$ then $[K : \mathbb{Q}] < \infty$.
  (b) If $[K : \mathbb{Q}] = \infty$, show that $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}(\alpha^2)$.

(2) Let G be a finite subgroup of the group $\mathrm{GL}_n(\mathbb{Q})$ of invertible n-by-n matrices with rational coefficients. Prove that every prime p which divides the order of G must satisfy $p \leq n + 1$.

(3) Let $R := K[X, Y]$ be the polynomial ring in two variables over the field K. Show that the ideal $M := \langle X, Y \rangle$ of R can be written as the union of prime ideals of R which are properly contained in M.

(4) Let H and J be subgroups of the finite group G such that the indices $[G : H]$ and $[G : J]$ are coprime. Show that every element of G can be written as hj for some $h \in H$ and $j \in J$.

(5) Show that the tensor product of $\mathbb{Z}$-modules $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.