QR Algebra
September 2016
Morning

(1) Suppose that $L \subseteq \mathbb{Z}^2$ is the subgroup generated by $(5,4)$ and $(2,7)$. Show that there is a unique subgroup $M \subseteq \mathbb{Z}^2$ of index 9 that contains L. Give generators of M.

(2) Suppose that A is an invertible square matrix with complex entries. Show that if A^2 is diagonalizable, then so is A.

(3) Suppose that R is the subring of the polynomial ring $\mathbb{Z}[x]$ consisting of all polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ for which the coefficients $a_1, a_2, \ldots, a_n$ are even (but a_0 does not have to be even).
  (a) Show that R contains a maximal ideal that is not finitely generated.
  (b) What is the ring $R/(3)$? Is it finite?

(4) Consider the vector space $V = \mathbb{F}_p^4$ where $\mathbb{F}_p$ is the field with p-elements.
  (a) How many 2-dimensional subspaces does V have?
  (b) Suppose that a subgroup $G \subset \mathrm{GL}_4(\mathbb{F}_p)$ is a p-group. Show that there exists a 2-dimensional subspace W of V such that $g \cdot W \subseteq W$ for all $g \in G$.

(5) Let K be the splitting field of $X^4 - 2$ over $\mathbb{Q}$.
  (a) What is the Galois group of K over $\mathbb{Q}$?
  (b) Find all subfields L of K such that $[L : \mathbb{Q}] = 4$. (Here $[L : \mathbb{Q}]$ is the degree of the field extension $L/\mathbb{Q}$.)

QR Algebra
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Afternoon

(1) Suppose that M is a field containing $\mathbb{F}_p$ and K and L are subfields of M. Assume that the number of elements of K, L and M are with p^6, p^{10} and p^{60} respectively. How many elements do the fields KL and $K \cap L$ have?

(2) Let $A \in \mathrm{Mat}_{n,n}(K)$ be an $n \times n$ matrix with entries in the field K and suppose that the characteristic polynomial of A is irreducible over K.
   (a) Let $L \subseteq \mathrm{Mat}_{n,n}(K)$ be the K-span of $I, A, A^2, \ldots, A^{n-1}$. Show that L is a subring of $\mathrm{Mat}_{n,n}(K)$, and show that the ring L is a field.
   (b) For a nonzero vector $v \in K^n$, prove that $v, Av, \ldots, A^{n-1}v$ form a basis of K^n.

(3) Calculate the following groups:
   (a) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(2)$.
   (b) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(9)$.
   (c) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(4) Let V be a vector space of dimension 3.
   (a) Show that there exists a linear map $\varphi : \bigwedge^2 V \otimes V \to \bigwedge^2 V \otimes V$ such that
$$\varphi((a \wedge b) \otimes c) = (a \wedge c) \otimes b - (b \wedge c) \otimes a.$$
   (b) Determine the eigenvalues of φ and their multiplicities.

(5) Let G be a group of even order. Show that there exists an element in G of order 2 whose conjugacy class has an odd number of elements.