1. Let
$$\alpha_1, \alpha_2, \ldots$$
be an enumeration of rational numbers in the interval $[0,1)$. Define $f : [0,1] \to \mathbb{R}$ as
$$f(x) = \sum_{\left\{j \middle| \alpha_j < x\right\}} \frac{1}{2^j}$$
for $0 < x \leq 1$ and $f(x) = 0$ for $x = 0$.
   a) Prove that f is discontinuous at rational x with $x < 1$.
   b) Prove that f is continuous at irrational x.

2. Let γ_n be the path
$$\gamma_n(t) = \left(t, \frac{\sin 2\pi t}{n}\right), \quad 0 \leq t \leq 1,$$
and let γ be the path $\gamma(t) = (t, 0)$, $0 \leq t \leq 1$, in the x-y plane. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, prove that
$$\lim_{n \to \infty} \int_{\gamma_n} f = \int_{\gamma} f,$$
where the integrals over γ and γ_n are path integrals.

3. Let $\omega = e^{\frac{2\pi i}{n}}$ with $n \in \mathbb{Z}$ and $n > 2$. Evaluate
$$1 + \omega^2 + \omega^4 + \cdots + \omega^{2(n-1)}.$$

4. Let $f(z)$ be defined of $\mathrm{Im} z \geq 0$ such that f is analytic for $\mathrm{Im} z > 0$, f is real if z is real, and f is continuous for all $\mathrm{Im} z \geq 0$. Define
$$f(z) = \overline{f(\bar{z})}$$
for Im z<0.
   a) Verify that f is differentiable for $\mathrm{Im} z < 0$.
   b) Verify that f is continuous for $\mathrm{Im} z = 0$.

5. Evaluate $\int_0^\infty \frac{dx}{x^{1/5}(1+x)}$ using complex integration.

Problem 1

(a) Suppose $x = \alpha_j$

and

$$\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3}, \ldots$$

to be an enumeration of rationals $< x$.

Then $f(x) = \sum_{i=1}^{\infty} \frac{1}{2^{k_i}}$

and

$$f(x+\epsilon) > f(x) + \frac{1}{2^j}$$

for any $\epsilon > 0$. Thus f is discont at x.

(b) Suppose x is irrational.

Given $\epsilon > 0$ choose J so large that

$$\sum_{j > J} \frac{1}{2^j} = \frac{1}{2^J} < \epsilon.$$

$$\delta = \frac{1}{2} \min_{j=1\ldots J} |\alpha_j - x|.$$

if $|x-y| < \delta$ then

$$x > \alpha_j \quad \text{for } j = 1 \ldots J$$

iff

$$y > \alpha_j.$$

Proof: $x > \alpha_j$ then $x > \alpha_j + \delta$

by choice of δ.

$$y - \alpha_j = (y-x) + (x - \alpha_j)$$

$$> (y-x) + \delta > 0.$$

Similarly, if $x < \alpha_j$ then $x + \delta < \alpha_j$.

$$\alpha_j - y = \alpha_j - x + x - y$$

$$> \alpha_j - x + \delta > 0.$$

Thus

$$|f(x) - f(y)| \le \sum_{j > J} \frac{1}{2^j} = \epsilon.$$

Problem 2

$$|\gamma_n - \gamma| = \left| \left(0, \frac{\sin 2\pi t}{n}\right) \right|$$

$$\leq \frac{1}{n} \quad \text{for} \quad 0 \leq t \leq 1.$$

$$\dot{\gamma}(t) = (1, 0)$$

$$\dot{\gamma}_n(t) = \left(1, \frac{2\pi \cos 2\pi t}{n}\right)$$

$$|\dot{\gamma}(t)| = 1$$

$$|\dot{\gamma}_n(t)| = \left(1 + \frac{4\pi^2 \cos^2 2\pi t}{n^2}\right)^{1/2}$$

$$0 < |\dot{\gamma}_n(t)| - |\dot{\gamma}(t)| \leq \frac{4\pi^2}{n^2}$$

because :

$$1 + x < 1 + 2x + x^2 \quad \text{for } x > 0$$

$$\Rightarrow$$

$$\sqrt{1+x} \leq 1 + x \quad \text{for } x \geq 0.$$

$$\int f(\gamma)\, |\dot{\gamma}|\, dt - \int f(\gamma_n)\, |\dot{\gamma}_n|\, dt$$

$$= \int (f(\gamma) - f(\gamma_n))\, |\dot{\gamma}|\, dt + \int f(\gamma_n)\, (|\dot{\gamma}| - |\dot{\gamma}_n|)\, dt$$

$$\int f(\gamma)\,|\dot\gamma|\,dt - \int f(\gamma_n)\,|\dot\gamma_n|\,dt$$

$$= \int (f - f(\gamma_n))\,|\dot\gamma|\,dt + \int f(\gamma_n)\,(|\dot\gamma| - |\dot\gamma_n|)\,dt$$

Therefore

$$\left| \int_\gamma f - \int_{\gamma_n} f \right|$$

$$\leq \int_0^1 |f(\gamma(t)) - f(\gamma_n(t))| \left|\frac{d\gamma}{dt}\right| dt$$

$$+ \int_0^1 |f(\gamma_n(t))| \left(\left|\frac{d\gamma_n}{dt}\right| - \left|\frac{d\gamma}{dt}\right| \right) dt.$$

FIRST $|\gamma - \gamma_n| \leq \frac{1}{n} \Rightarrow |f(\gamma(t)) - f(\gamma_n(t))| \to 0$

as $n \to \infty$ unif in t

because f is unif contin.

SECOND γ_n, γ are contained in

$$|x| \leq 2, \quad |y| \leq 2$$

and we may assume $|f| \leq M$ in this region.

We verified:
$$\left|\frac{d\gamma_n}{dt}\right| - \left|\frac{d\gamma_n}{dt}\right| \leq \frac{4\pi^2}{n^2}$$

Therefore both terms in the upper bound $\to 0$ as $n \to \infty$.

Problem 3

$$\omega = e^{2\pi i/n}$$

Then $\omega^n = 1$ and $(\omega^2)^n = 1$

$$z^n - 1 = (z-1)(1 + z + \cdots + z^{n-1})$$

$$\omega^{2n} - 1 = (\omega^2 - 1)(1 + \omega^2 + \cdots + \omega^{2(n-1)})$$

$$(\omega^2 - 1)\,(1 + \omega^2 + \cdots + \omega^{2(n-1)}) = 0$$

If $n > 2$, $\omega^2 \neq 1$.

Thus $1 + \omega^2 + \cdots + \omega^{2(n-1)} = 0$.

Problem 4

(b) If $\mathrm{Im}(z) < 0$ and $z \to a$ with a real

$$\lim_{z \to a} f(z) = \lim_{z \to a} \overline{f(\bar{z})} = \overline{f(a)} = f(a)$$

(a) Assume $\mathrm{Im}\, z < 0$.

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{f(\bar{z} + \bar{h})} - \overline{f(\bar{z})}}{h}$$

for h small.

$$= \overline{\left(\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}\right)}$$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

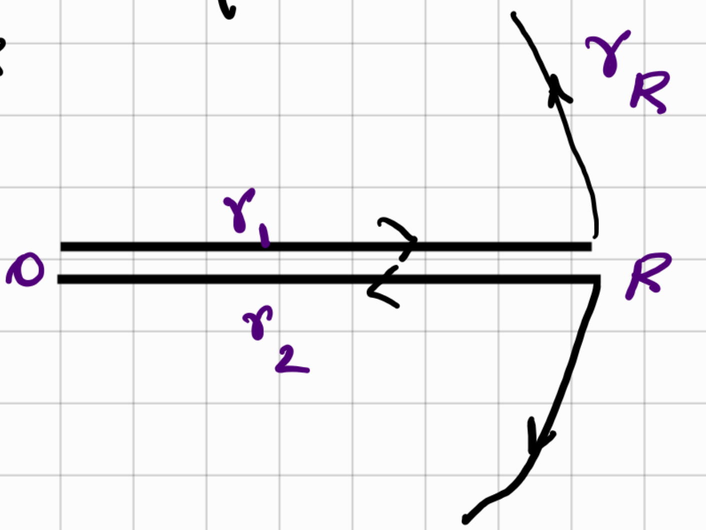
$$= \lim_{h \to 0} \overline{\left(\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}\right)}$$

$$= \overline{f'(\bar{z})}$$ because f is analytic at $\bar{z}$.

Problem 5

consider $$\int_{\gamma_1 \cup \gamma_2 \cup \gamma_R} \frac{dz}{z^{1/5}(1+z)}$$

with (



The branch $z^{1/5} = r^{1/5} e^{\frac{i\theta}{5}}$

with $0 < \theta < 2\pi$.

+ve real line is the branch cut.

γ_1 is slightly above, γ_2 slightly below.

$$\int_{\gamma_1 \cup \gamma_2 \cup \gamma_R} \frac{dz}{z^{1/5}(1+z)} = 2\pi i \, \mathrm{Res}\left(\frac{1}{z^{1/5}(1+z)} ; -1 \right)$$

$$= \frac{2\pi i}{e^{i\pi/5}}$$

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{dz}{z^{1/5}(1+z)} = 0$$

$$\lim_{R \to \infty} \int_{\gamma_1} \frac{dz}{z^{1/5}(1+z)} = \int_0^\infty \frac{dx}{x^{1/5}(1+x)}$$

$$\lim_{R\to\infty} \int_{\gamma_2} \frac{dz}{z^{1/5}(1+z)}$$

$$= \int_{\infty}^{0} \frac{dx}{x^{1/5} e^{\frac{2\pi i}{5}} (1+x)}$$

$$= \frac{-1}{e^{2\pi i/5}} \int_0^{\infty} \frac{dx}{x^{1/5}(1+x)}$$

Thus $\displaystyle\int_0^{\infty} \frac{dx}{x^{1/5}(1+x)} = \frac{2\pi i}{e^{i\pi/5}} \frac{e^{2\pi i/5}}{(e^{2\pi i/5}-1)}$

$$= \pi \frac{(2i)}{(e^{i\pi/5} - e^{-i\pi/5})}$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{5}\right)}.$$