AIM Qualifying Review Exam: Probability and Discrete Mathematics

January 7, 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

A deck of cards consists of four suits (clubs, diamonds, hearts, spades) and thirteen cards in each suit. What is the probability that a hand of five cards chosen from deck randomly without replacement contains at least one card of each suit?

Solution

Ross 11, p. 57.

Note that we are asking about getting exactly 2 cards from one suit and exactly one card from each of the other suits. We'll take the cards in order.

There are $\binom{5}{2} = 10$ ways to pick the positions of the repeated suit. Then there are $4! = 24$ ways to assign the first appearances of the four suits. Once the suit configuration has been assigned, there are $13^4 \cdot 12$ ways to pick cards with prescribed suits.

To get the probability, divide by the denominator of the total number of ways to draw five cards in order, and get $\frac{\binom{5}{2}4!13^4 \cdot 12}{52 \cdot 51 \cdots 48}$.

Mathematical concepts: discrete random variables, counting, combinations, permutations, binomial coefficient, multinomial coefficient

Problem 2

Suppose that continuous random variable X has cumulative distribution function $F(x)$. What is the cumulative distribution function of e^X?

Solution

Ross, p. 180, 4.2.

We are told $\Pr(X \leq a) = F(a)$. So $\Pr(e^X \leq b) = \Pr(X \leq \log(b)) = F(\log(b))$. More precisely, the distribution function is $F(\log(x))$ for $x > 0$ and 0 for $x \leq 0$.

Mathematical concepts: Continuous random variables, distribution functions

Problem 3

Determine the number of ways to color a 1-by-n chessboard with maize, blue, and khaki such that the number of maize squares is even and there is at least one blue square. *Hint: Try an exponential generating function.*

Solution

Brualdi example, page 227.

The exponential generating function is the product of exponential generating functions for the three colors:

$$\begin{aligned}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right) &= \frac{e^x+e^{-x}}{2}e^x(e^x-1)\\ &= \frac{e^{3x}-e^{2x}+e^x-1}{2}.\end{aligned}$$

We want h_n, where the coefficients are $\frac{h_n}{n!}=\frac{3^n-2^n+1}{2\cdot n!}$ for $n>0$ and $h_0=0$.

Mathematical concepts: exponential generating function

Problem 4

Recall that a partial order consists of a set of elements and a relation $\leq$ that is transitive and reflexive (but some pairs of elements may be not be comparable to each other). There are special functions on pairs of elements in the partial order:

$$\delta(x,y)=\begin{cases}1, & x=y;\\ 0, & x\neq y,\end{cases}$$

and $\mu(x,y)$, defined for $x\leq y$ by

$$\sum_{z:\,x\leq z\leq y}\mu(x,z)=\delta(x,y).$$

(a) Find $\mu(x,y)$ for the linear order $0\leq 1\leq 2\cdots\leq n-1$.

(b) Show $\mu(A,B)=(-1)^{|B|-|A|}$ if $A\subseteq B$ are sets ordered by subset inclusion $\subseteq$.

Solution

Brualdi, pp. 187–188

For the linear order, $\mu(x,x)=1$ since there is only one term in the sum $\sum\mu(x,z)=\delta(x,x)=1$. Therefore, $\mu(x,x)+\mu(x,x+1)=\delta(x,x+1)=0$ so $\mu(x,x+1)=-1$. All other values of δ are zero and that implies that all other values of μ are zero.

For set inclusion with $A\subseteq B$, we have $\mu(A,B)=1$ if $A=B$, as above. By induction, suppose $A\subsetneq B$, so $\delta(A,B)=0$. We have

$$\begin{aligned}\mu(A,B) &= -\sum_{A\subseteq C\subsetneq B}\mu(A,C)\\ &= -\sum(-1)^{|C|-|A|}\\ &= -\sum_{k=0}^{p-1}(-1)^k\binom{p}{k},\end{aligned}$$

where $p = |B \setminus A|$, and where we've used the fact that there are $\binom{p}{k}$ ways to pick k elements from $B \setminus A$ to add to A to get a C.

The binomial theorem gives

$$0 = (1-1)^n = \sum_{k=0}^{p} (-1)^k \binom{p}{k},$$

so the above sum becomes $-\sum_{k=0}^{p-1}(-1)^k\binom{p}{k} = -(-1)^p\binom{p}{p}$. Canceling the negative sign, we get $\mu(A,B) = (-1)^p\binom{p}{p} = (-1)^{|B|-|A|}$.

Mathematical concepts: Partial orders, Moebius inversion

<u>Problem 5</u> An undirected weighted graph consists of a set of vertices and a weight w_e on each edge

(unordered pair of vertices). A minimum spanning tree is a connected subgraph (some edges removed) in which there are no cycles and the sum of the weights on remaining edges is minimized.

Suppose all edge weights are distinct. Show that minimum spanning trees are unique.

Solution

Kleinberg and Tardos, p 192, number 8.

Suppose T and T' are two different minimum spanning trees. Remove the most expensive edge e in the set difference $T \Delta T'$; suppose it is in $T \setminus T'$. So $T - e$ consists of two connected components and some edge e' in T' reconnects these components (since T' is connected—so there's a path in T' between the endpoints of e and a first edge along that path with endpoints in different components of $T - e$). By distinctness of the weights and maximality of e, we have $w_{e'} < w_e$. So $T - e + e'$ is a spanning tree and is less costly than T.

Mathematical concepts: Graph algorithms, minimality