AIM Qualifying Review Exam: Probability and Discrete Mathematics

January 7, 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

A deck of cards consists of four suits (clubs, diamonds, hearts, spades) and thirteen cards in each suit. What is the probability that a hand of five cards chosen from deck randomly without replacement contains at least one card of each suit?

Problem 1

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Problem 2

Suppose that continuous random variable X has cumulative distribution function $F(x)$. What is the cumulative distribution function of e^X?

Problem 2

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Problem 3

Determine the number of ways to color a 1-by-n chessboard with maize, blue, and khaki such that the number of maize squares is even and there is at least one blue square. *Hint: Try an exponential generating function.*

Problem 3

Problem 3

Problem 3

Problem 4

Recall that a partial order consists of a set of elements and a relation $\leq$ that is transitive and reflexive (but some pairs of elements may be not be comparable to each other). There are special functions on pairs of elements in the partial order:

$$\delta(x, y) = \begin{cases} 1, & x = y; \\ 0, & x \neq y, \end{cases}$$

and $\mu(x, y)$, defined for $x \leq y$ by

$$\sum_{z\colon x \leq z \leq y} \mu(x, z) = \delta(x, y).$$

(a) Find $\mu(x, y)$ for the linear order $0 \leq 1 \leq 2 \cdots \leq n - 1$.

(b) Show $\mu(A, B) = (-1)^{|B|-|A|}$ if $A \subseteq B$ are sets ordered by subset inclusion $\subseteq$.

Problem 4

Problem 4

Problem 4

Problem 5 An undirected weighted graph consists of a set of vertices and a weight w_e on each edge (unordered pair of vertices). A minimum spanning tree is a connected subgraph (some edges removed) in which there are no cycles and the sum of the weights on remaining edges is minimized.

Suppose all edge weights are distinct. Show that minimum spanning trees are unique.

Problem 5

Problem 5

Problem 5