AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) (5 points) Let $\{\mathbf{a}, \mathbf{b}\}$ be a basis for $\mathbb{R}^2$ and $\mathbf{A}$ be a 2-by-2 matrix such that $\mathbf{Aa} = \mathbf{b}$ and $\mathbf{Ab} = \mathbf{a}$.

Find the eigenvalues and eigenvectors of $\mathbf{A}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) (5 points) Show that if $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis then $\|\mathbf{A}\|_2 = 1$.

(c) (10 points) Let $\{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ be a basis for $\mathbb{R}^4$ and $\mathbf{B}$ be a 4-by-4 matrix such that $\mathbf{Bc} = \mathbf{d}$, $\mathbf{Bd} = \mathbf{e}$, $\mathbf{Be} = \mathbf{f}$, and $\mathbf{Bf} = \mathbf{c}$. Find the eigenvalues and the determinant of $\mathbf{B}$.

Solution

(a) $\mathbf{a} \pm \mathbf{b}$ are eigenvectors with eigenvalues ± 1.

(b) Let $\mathbf{V}$ be the matrix with columns $\{(\mathbf{a}+\mathbf{b})/\sqrt{2}, (\mathbf{a}-\mathbf{b})/\sqrt{2}\}$, which is also an orthonormal basis if $\{\mathbf{a}, \mathbf{b}\}$ is. Then $\mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{D}^2\mathbf{V}^{-1}$ where $\mathbf{D}$ is diagonal with 1 and -1 on the main diagonal. Hence the singular values of $\mathbf{A}$, i.e the square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$, are both 1.

(c) Let $\mathbf{V}$ be the matrix with columns $\{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$. In the basis of $\mathbf{V}$, $\mathbf{B}$ becomes

$$\mathbf{C} = \mathbf{V}\mathbf{B}\mathbf{V}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and it has the same eigenvalues and determinant as $\mathbf{B}$. Using row operations we can compute $\det(\lambda\mathbf{I} - \mathbf{C})$ by reducing it to a diagonal matrix whose determinant is the product of the diagonal entries, $\lambda^4 + 1$. The eigenvalues are thus the fourth roots of unity: $\pm 1, \pm i$. The determinant is their product, -1.

Problem 2

(a) (8 points) Consider the subspace S of $\mathbb{R}^3$ given by $x + 2y + 3z = 0$ for $[x, y, z]^T \in \mathbb{R}^3$. Let $\mathbf{M}$ be the matrix that reflects $\mathbb{R}^3$ through S. I.e. $\mathbf{Mu} = \mathbf{u}$ for $\mathbf{u} \in S$ and $\mathbf{Mv} = -\mathbf{v}$ for $\mathbf{v} \in S^\perp$, the orthogonal complement of S. Write $\mathbf{M}$ explicitly, i.e. all of its entries.

(b) (4 points) Prove or disprove: A projection matrix (i.e. a matrix $\mathbf{P}$ such that $\mathbf{P}^2 = \mathbf{P}$) may have an eigenvalue greater than 1.

(c) (8 points) Prove or disprove: A projection matrix may have a singular value greater than 1.

Solution

(a) Let $\mathbf{P}$ be the orthogonal projection onto $S^\perp$. Then $\mathbf{M} = \mathbf{I} - 2\mathbf{P}$. We have

$$\mathbf{P} = \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad ; \quad \mathbf{M} = \frac{1}{7}\begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix}$$

(b) This is false. If λ is an eigenvalue, $\lambda\mathbf{v} = \mathbf{Pv} = \mathbf{P^2v} = \lambda^2\mathbf{v}$, so λ is 0 or 1.

(c) This is true if we have an oblique projection matrix. The largest singular value of a projection matrix $\mathbf{P}$ is $\|\mathbf{P}\|_2 = \sup_{\mathbf{v}}\|\mathbf{Pv}\|_2/\|\mathbf{v}\|_2$. Consider the rank-one projection $\mathbf{P} = \mathbf{u}\mathbf{w}^T/\mathbf{w}^T\mathbf{u}$ where $\mathbf{w}^T\mathbf{u} \neq 0$. The range of $\mathbf{P}$ is the span of $\mathbf{u}$ and the null space is the orthogonal complement to the span of $\mathbf{w}$. $\sup_{\mathbf{v}}\|\mathbf{Pv}\|_2/\|\mathbf{v}\|_2 \geq \|\mathbf{Pw}\|_2/\|\mathbf{w}\|_2 = \|\mathbf{u}\|_2\|\mathbf{w}\|_2/|\mathbf{w}^T\mathbf{u}|$. So $\mathbf{P}$ maps $\mathbf{w}$ to a longer vector ($\|\mathbf{P}\|_2 > 1$) as long as $\mathbf{w}$ is not aligned with $\mathbf{u}$.

Problem 3

Consider the differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx + x^7 = 0. \tag{1}$$

with x, t, b, and c real.

(a) (4 points) For $b > 0$, show that any solution to equation (1) remains bounded as $t \to +\infty$.

(b) (4 points) Let x_1 be a solution to equation (1) with $b = 0$ and $c > 0$. Let $x_1(0) = 0.1$ and $x_1'(0) = 0$. Let $x_1(t_0) = 0$ for some time t_0. What are the possible values of $x_1'(t_0)$?

(c) (4 points) Estimate t_0 from part b.

(d) (4 points) Prove that, with the initial conditions in part b, y_1 is the unique solution for some time interval.

(e) (4 points) Find the equilibria for $b = 0$ and $c < 0$. Using the concepts of kinetic and potential energy (without performing any detailed calculations), describe $x(t)$ for $t > 0$ given $x'(0) = 0$ and $x(0)$ that is slightly perturbed away from each equilibrium.

Solution

The equation describes a nonlinear spring with damping constant b and stiffness c (possibly negative) for the linear part of the spring force.

(a) We multiply by x' and obtain

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}cx^2 + \frac{1}{8}x^8\right) = -b\left(\frac{dx}{dt}\right)^2 < 0.$$

So $\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}cx^2 + \frac{1}{8}x^8\right)$ is bounded by its value at any finite time, and therefore $\left(\frac{1}{2}cx^2 + \frac{1}{8}x^8\right)$ is bounded by the same value. If $|x|$ grows without bound, $\left(\frac{1}{2}cx^2 + \frac{1}{8}x^8\right)$ does also, whether c is positive or negative, because of the dominant $x^8/8$ term. This is a contradiction, and therefore $|x|$ remains bounded.

(b) We have $\left(\frac{1}{2}x_1'^2 + \frac{1}{2}cx_1^2 + \frac{1}{8}x_1^8\right) = \text{constant} = \frac{c}{2} \times 10^{-2} + \frac{1}{8} \times 10^{-8} = \frac{1}{2}x_1'(t_0)^2$, so $x_1'(t_0) = \pm\sqrt{c \times 10^{-2} + \frac{1}{4} \times 10^{-8}}$.

(c) We estimate t_0 by neglecting the nonlinear term. This is reasonable since $|x_1| \leq 10^{-7}$. t_0 is shifted forward or backward from 0 by an integer multiple of a period plus or minus a quarter period (for the linear approximation). The period is $2\pi/\sqrt{c}$, so $t_0 \approx 2\pi(n \pm 1/4)/\sqrt{c}$ for some integer n.

(d) We write the equation as a first-order system $x' = y; y' = -cx - x^7$, in order to apply the nonlinear existence and uniqueness theorem. Since the right hand sides and their first partial derivatives with respect to x and y are continuous functions of x and y for all x and y, there is some interval of time about the initial condition in which we have a unique solution.

(e) The equilibria have $-cx - x^7 = 0$ and $y = x' = 0$. There are three real solutions: $x = 0, \pm(-c)^{1/6}$. $x = 0$ is a local maximum of potential energy, and $x = \pm(-c)^{1/6}$ are local minima. For $x(0)$ slightly perturbed away from $x = \pm(-c)^{1/6}$, $x(t)$ is a small periodic oscillation about these equilibria, which are stable. For $x(0)$ slightly perturbed away from $x = 0$, $x(t)$ oscillates periodically about whichever of $\pm(-c)^{1/6}$ $x(0)$ is closer to, such that the sum of kinetic and potential energy are conserved

Problem 4

(a) (10 points) Find the solution of the initial value problem

$$ty' + 2y = \frac{\cos t}{t},\ y(\pi/4) = 0.$$

(b) (10 points) Find the form of the general solution to the ODE

$$\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} = 5e^t + 2t^3e^{-t} + te^{-t}\sin t + e^{-t}\cos t.$$

Write your answer in the form of a linear combination of functions of t with all of the coefficients left undetermined.

Solution

(a) Since this is a linear first order equation, we can use the integrating factor method. To find the integrating factor we first divide both sides by t, and find the integrating factor as $e^{\int 2dt/t} = t^2$. We then multiply both sides by it to obtain

$$t^2y' + 2ty = (t^2y)' = \cos t$$

so

$$y = \frac{\sin t}{t^2} - \frac{1}{\sqrt{2}t^2},$$

with the constant in the second term fixed by the initial condition.

(b) We first find the solution to the homogeneous equation

$$\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} = 0$$

by inserting $y = e^{rt}$. We have $r = 0$ with multiplicity 2 and $-1 \pm i$. So the solution to the homogeneous equation is

$$y = A + Bt + Ce^{-t}\sin t + De^{-t}\cos t.$$

We then add terms for each type of exponential on the right hand side. If the exponential is muliplied by a polynomial in t we take an arbitrary polynomial in t of the same degree times that exponential. Then, if necessary, we multiply by the smallest power of t so that none of the terms overlap with those in the homogeneous solution. So for $5e^t$ we have Ee^t; for $2t^3e^{-t}$ we have $(Ft^3 + Gt^2 + Ht + I)e^{-t}$; for $te^{-t}\sin t + e^{-t}\cos t$ (both exponentials of type $e^{(-1\pm i)t}$) we have $t(Jt + K)e^{-t}\sin t + t(Lt + M)e^{-t}\cos t$, the extra power of t to avoid overlap with the homogeneous solution. The full solution is

$$y = A + Bt + Ce^{-t}\sin t + De^{-t}\cos t + Ee^t + (Ft^3 + Gt^2 + Ht + I)e^{-t}$$
$$+ t(Jt + K)e^{-t}\sin t + t(Lt + M)e^{-t}\cos t.$$

Problem 5

Solve the PDE

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u = 0$$

for $u(r,\theta)$ in the angular sector $\{r > 0\ ;\ -\pi/4 < \theta < \pi/4\}$ with the boundary conditions:

$$\frac{\partial u}{\partial\theta}(r, -\pi/4) = r^2, \quad u(r, \pi/4) = 1, \quad r > 0.$$

Solution

We plug in a separation of variables solution $u = R(r)Q(\theta)$ and obtain

$$\frac{r^2R''}{R} + \frac{rR'}{R} = \frac{-Q''}{Q} = \lambda^2.$$

which has general solution $R = r^{\pm\lambda}$ and $Q = A\sin\lambda\theta + B\cos\lambda\theta$. We can match the boundary conditions by superposing solutions with $\lambda = 0$ and 2: $u = A + Br^2\sin 2\theta + Cr^2\cos 2\theta$. The boundary condition at $-\pi/4$ gives $C = 1/2$, and the boundary condition at $\pi/4$ gives $A = 1$, $B = 0$. The final answer is $u = 1 + \frac{1}{2}r^2\cos 2\theta$.