AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) (5 points) Let $\{\mathbf{a}, \mathbf{b}\}$ be a basis for $\mathbb{R}^2$ and $\mathbf{A}$ be a 2-by-2 matrix such that $\mathbf{A}\mathbf{a} = \mathbf{b}$ and $\mathbf{A}\mathbf{b} = \mathbf{a}$.

Find the eigenvalues and eigenvectors of $\mathbf{A}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) (5 points) Show that if $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis then $\|\mathbf{A}\|_2 = 1$.

(c) (10 points) Let $\{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ be a basis for $\mathbb{R}^4$ and $\mathbf{B}$ be a 4-by-4 matrix such that $\mathbf{B}\mathbf{c} = \mathbf{d}$, $\mathbf{B}\mathbf{d} = \mathbf{e}$, $\mathbf{B}\mathbf{e} = \mathbf{f}$, and $\mathbf{B}\mathbf{f} = \mathbf{c}$. Find the eigenvalues and the determinant of $\mathbf{B}$.

Problem 1

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Problem 2

(a) (8 points) Consider the subspace S of $\mathbb{R}^3$ given by $x + 2y + 3z = 0$ for $[x, y, z]^T \in \mathbb{R}^3$. Let $\mathbf{M}$ be the matrix that reflects $\mathbb{R}^3$ through S. I.e. $\mathbf{Mu} = \mathbf{u}$ for $\mathbf{u} \in S$ and $\mathbf{Mv} = -\mathbf{v}$ for $\mathbf{v} \in S^\perp$, the orthogonal complement of S. Write $\mathbf{M}$ explicitly, i.e. all of its entries.

(b) (4 points) Prove or disprove: A projection matrix (i.e. a matrix $\mathbf{P}$ such that $\mathbf{P}^2 = \mathbf{P}$) may have an eigenvalue greater than 1.

(c) (8 points) Prove or disprove: A projection matrix may have a singular value greater than 1.

Problem 2

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Problem 3

Consider the differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx + x^7 = 0. \tag{1}$$

with x, t, b, and c real.

(a) (4 points) For $b > 0$, show that any solution to equation (1) remains bounded as $t \to +\infty$.

(b) (4 points) Let x_1 be a solution to equation (1) with $b = 0$ and $c > 0$. Let $x_1(0) = 0.1$ and $x_1'(0) = 0$. Let $x_1(t_0) = 0$ for some time t_0. What are the possible values of $x_1'(t_0)$?

(c) (4 points) Estimate t_0 from part b.

(d) (4 points) Prove that, with the initial conditions in part b, y_1 is the unique solution for some time interval.

(e) (4 points) Find the equilibria for $b = 0$ and $c < 0$. Using the concepts of kinetic and potential energy (without performing any detailed calculations), describe $x(t)$ for $t > 0$ given $x'(0) = 0$ and $x(0)$ that is slightly perturbed away from each equilibrium.

Problem 3

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Problem 4

(a) (10 points) Find the solution of the initial value problem

$$ty' + 2y = \frac{\cos t}{t}, \; y(\pi/4) = 0.$$

(b) (10 points) Find the form of the general solution to the ODE

$$\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} = 5e^t + 2t^3e^{-t} + te^{-t}\sin t + e^{-t}\cos t.$$

Write your answer in the form of a linear combination of functions of t with all of the coefficients left undetermined.

Problem 4

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Problem 5

Solve the PDE

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) u = 0$$

for $u(r, \theta)$ in the angular sector $\{r > 0\ ;\ -\pi/4 < \theta < \pi/4\}$ with the boundary conditions:

$$\frac{\partial u}{\partial \theta}(r, -\pi/4) = r^2, \quad u(r, \pi/4) = 1, \quad r > 0.$$

Problem 5

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