Problem 1

$$f(0) = f(1 + (-1)) = f(1)\, f(-1)$$

Therefore $f(-1) = 1/2$.

if $n \in \mathbb{Z}^+$ $f(n) = f(1 + \cdots + 1)$

$$= f(1)^n$$

$$= 2^n$$

and

$$f(-n) = f(-1 - 1 \cdots - 1)$$

$$= f(-1)^n$$

$$= 2^{-n}$$

Thus $f(n) = 2^n$ for all $n \in \mathbb{Z}$.

If p/q is a rational number w/ $q \in \mathbb{Z}^+$.
then

$$f(p) = f(p/q + \cdots + p/q)$$

$$= f(p/q)^q$$

$\Rightarrow$

$$f\left(\frac{p}{q}\right) = 2^{p/q}.$$

Thus we have proved that $f(x) = 2^x$ for all $x \in \mathbb{Q}$.

Now consider $x \in \mathbb{R}$.
If f is upper semicontinuous

$$U = \{ y \in \mathbb{R} \mid f(y) < 2^x \}$$

must be open.

If $p \in \mathbb{Q}$ and $p < x$ then $p \in U$ because $2^p < 2^x$.

OTOH if $q \in \mathbb{Q}$ and $x < q$ then $q \notin U$.

$U^c = \mathbb{R} - U$ must be closed.

Thus for any real $y \geq x$, we must have

$$f(y) \geq 2^x.$$

In particular, $f(x) \geq 2^x$ for all x.

In fact one may prove $f(x) = 2^x$ for all x.

Problem 2

(a) If $f(x) = \begin{cases} 1/n & \text{for } x = q_n \\ 0 & \text{otherwise} \end{cases}$

then $\int_0^1 f(x)\,dx = 0.$

if $0 = x_0 < x_1 < \cdots < x_N = 1$ divides $[0,1]$ into segments with

$$x_j - x_{j-1} < \epsilon$$

for $j = 1 \cdots N$, then

$$\sum_{1}^{N} f(\tilde{x}_j)(x_j - x_{j-1}) < \frac{1}{n} + \epsilon \cdot n$$

because

1. intvls which contain one of $q_1 \cdots q_n$ contribute at most ϵn.

2. intvls which do not contain $q_1 \cdots q_n$ contribute at most $\frac{1}{n}$.

The proof may be completed by setting $\epsilon = 1/n^2$ and taking $n \to \infty$.

(b) $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$

$= 0$ otherwise.

Problem 3

if $z_1 = z_2 = e^{-2\pi i/3}$ then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) + 2\pi i$$

In general, if $\arg(z_1) + \arg(z_2) < -\pi$ then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) + 2\pi i$$

if $z_1 = z_2 = 1$ then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2).$$

In general,

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

if $-\pi < \arg(z_1) + \arg(z_2) < \pi.$

if $z_1 = z_2 = e^{2\pi i/3}$ then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) - 2\pi i.$$

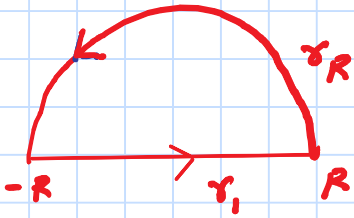
In general,

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) - 2\pi i.$$

if $\arg(z_1) + \arg(z_2) > \pi.$

Problem 4

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\log|x|}{1+x^2}\,dx.$$



For the principal br of $\log z$,

$$\int_{\gamma_1 \cup \gamma_R} \frac{\log z}{1+z^2}\,dz = 2\pi i \cdot \operatorname{Res}\left(\frac{\log z}{(z+i)(z-i)}; i\right)$$

$$= 2\pi i \cdot \frac{\log i}{2i}$$

$$= \pi\left(\frac{i\pi}{2}\right) = i\frac{\pi^2}{2}$$

$$\left|\int_{\gamma_R} \frac{\log z}{1+z^2}\,dz\right| \leq \int_{\gamma_R} \frac{|\log z|}{|1+z^2|}\,|dz|$$

$$\leq 2\pi R \frac{(\log R + \pi)}{R^2 - 1}$$

$$\to 0 \text{ as } R \to \infty.$$

$$\int_{-\infty}^{\infty} \frac{\log|x|}{1+x^2}\,dx = \lim_{R\to\infty} \operatorname{Re} \int_{\gamma_1 \cup \gamma_R} \frac{\log z}{1+z^2}\,dz$$

$$= \operatorname{Re}\frac{i\pi^2}{2} = 0$$

Problem 5

(a) $f(z) = \frac{1}{z^2+2}$

has poles at $\pm\sqrt{2}\,i$ both of them outside γ.

Therefore, the answer is 0.

(b) $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

has no poles and all zeros are on the real line.

The only zero inside γ is $z=0$.

The answer is 1.

(c) $\tan w = \frac{1}{i}\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$

All zeros and poles of $\tan w$ are on the real line.

$2z^2 = 0$, $z=0$ is the only soln of

$\tan(2z^2) = 0$

inside γ.

it is a double zero.

The poles inside γ are given by

$$2z^2 = \pm \pi/2$$

or

$$z = +\frac{\sqrt{\pi}}{2}, \ -\frac{\sqrt{\pi}}{2}, \ i\frac{\sqrt{\pi}}{2}, \ -i\frac{\sqrt{\pi}}{2}.$$

To figure out the order of the pole at $z = \sqrt{\pi}/2$, set

$$z = \frac{\sqrt{\pi}}{2} + w.$$

Then $\tan 2z^2 = \tan\left(\frac{\pi}{2} + 2\sqrt{\pi}w + 2w^2\right)$

$$= -\frac{\cos(2\sqrt{\pi}\,w + 2w^2)}{\sin(2\sqrt{\pi}w + 2w^2)}$$

This is a simple pole because

$$\lim_{w \to 0} -w\frac{\cos(2\sqrt{\pi}\,w + 2w^2)}{\sin(2\sqrt{\pi}\,w + 2w^2)} = -\frac{1}{2\sqrt{\pi}}.$$

Similarly the other three poles are also simple.

Answer $= 2 - 4 = -2$.