AIM Qualifying Exam: Advanced Calculus and Complex Variables

Jan 2023

There are five (5) problems in this test, each worth 20 points.

For the most part, there is sufficient room in this booklet for all your work. However, if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a function such that

$$f(x+y) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$ and $f(0) = 1, f(1) = 2$.

a) Prove that $f(n) = 2^n$ for all integers n.

b) A function $g : \mathbb{R} \to \mathbb{R}^+$ is upper semicontinuous if

$$\limsup_{n\to\infty} g(x_n) <= g(x)$$

whenever $\lim_{n\to\infty} x_n = x$. If f is upper semicontinuous, prove that $f(x) = 2^x$ for all $x \in \mathbb{R}$.

2. Let $q_1, q_2, \ldots$ be an enumeration of rational numbers in $[0, 1]$. Define a function f such that

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = q_n \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

a) What is the Riemann integral $\int_0^1 f(x)\,dx$? Justify your answer.

b) Give an example of a function $f : [0, 1] \to [0, 1]$ whose Riemann integral does not exist.

3. For $z \neq 0$ and z non-negative, define

$$\log z = \log|z| + i\arg(z)$$

with $\arg(z)$ chosen in the interval $(-\pi, \pi)$. This is the principal branch of the logarithm. We may then have

$$\log z_1 z_2 = \begin{cases} \log z_1 + \log z_2 - 2\pi i \\ \log z_1 + \log z_2 \\ \log z_1 + \log z_2 + 2\pi i \end{cases}$$

where $z_1, z_2 \in \mathbb{C}$ with both nonzero and non-negative.

a) Give examples of z_1, z_2 for each of the three cases.

b) Completely describe when each case applies.

4. Use complex integration to evaluate

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx.$$

5. The winding number of a closed curve γ in $\mathbb{C}$ that does not pass through zero is the number of times γ circles 0 in the counter-clockwise sense. If γ is the curve $\gamma(t) = e^{2\pi it}$, $0 \le t \le 1$, find the winding number of $f(\gamma(t))$, $0 \le t \le 1$, for each f below:

   a) $f(z) = \frac{1}{z^2+2}$

   b) $f(z) = \sin z$

   c) $f(z) = \tan(2z^2)$.