AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2020

There are five (5) problems in this examination.

You may write your answers on these pages or separate paper. Be sure to put your secret number in the top right corner of each page. When you are finished, write the number of each page of out the total number of pages i.e. 1/6, 2/6 etc.

No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Let the matrix $\mathbf{A}$ satisfy the two equations $\mathbf{A}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$. What are the possible values of the rank and nullity of $\mathbf{A}$? Justify your answer.

(b) Let the matrix $\mathbf{B}$ satisfy the three equations $\mathbf{B}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{B}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, and $\mathbf{B}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$. Find the null space of $\mathbf{B}$.

(c) Consider all matrices $\mathbf{C}$ such that $\mathbf{BC}$ is a scalar multiple of the identity matrix. Show that the set of such matrices $\{\mathbf{C}\}$ is a linear space, and find its dimension.

Problem 2

(a) Let the matrix $\mathbf{A}$ have the following property: for any two eigenvectors v_1 and v_2 of $\mathbf{A}$, $v_1 + v_2$ is also an eigenvector of $\mathbf{A}$. How many distinct eigenvalues could $\mathbf{A}$ have? Justify your answer.

(b) Is there a matrix $\mathbf{M}$ with complex entries—i.e. with nonzero imaginary part—but only real eigenvalues? Justify your answer.

(c) Is there a real 100-by-100 matrix $\mathbf{C}$ that has 99 nonzero imaginary eigenvalues and one real eigenvalue? Justify your answer.

(d) Let $\mathbf{N}$ be a purely imaginary matrix—i.e. every entry of $\mathbf{N}$ has zero real part. What are the possible values of the determinant of $\mathbf{N}$? Justify your answer.

Problem 3 Find all real choices of the constants a, b, and c such that the solutions of

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = \sin(ct)$$

remain bounded in time for all initial conditions.

Problem 4

Consider

$$(\tan t)\frac{d^2y}{dt^2} - (\sin t)\frac{dy}{dt} + |\log t|^{10^{1000}} y = 0. \tag{1}$$

(a) Find an expression for $y_1(t)y_2'(t) - y_2(t)y_1'(t)$ that is correct up to a multiplicative constant for any two solutions $y_1(t)$ and $y_2(t)$.

(b) Assume we have a solution $y(t)$ such that $y(5)$ and $y'(5)$ are two given constants. For which intervals of t can you guarantee $y(t)$ remains bounded?

(c) Find the set of initial conditions for which there is a solution to:

$$(y'')^{2000} + \log(|\cos y|) = y'. \tag{2}$$

Problem 5

Solve the equation

$$\partial_{xx}u + \partial_{yy}u = 0 \ , \ 0 < x < a, \ 0 < y < 1$$

with the conditions:

$$u(x,0) = 0 \ , u(x,1) = f(x) \ , \ 0 \leq x \leq a$$
$$\partial_x u(0,y) = \partial_x u(a,y) = 0 \ \ , \ 0 \leq y \leq 1.$$