AIM Qualifying Exam: Advanced Calculus and Complex Variables
August 2020
For full credit, support your answers with appropriate explanations.
There are five problems, each worth 20 points.

1. (20 points) A map from the z-plane to the w-plane given by

$$w = \frac{az+b}{cz+d}$$

is called a fractional linear transformation if $ad - bc \neq 0$.

   (a) (10 points) Find a fractional linear transformation that maps $\Im z > 0$ to $|w| < 1$.

   (b) (10 points) Find a fractional linear transformation that maps $|z| < 1$ to $|w| < 1$ such that $z = \frac{1}{2}$ maps to $w = 0$.

2. Consider the polynomial $p(z) = z^3 + 8z + 1$.

   (a) (10 points) Prove that all roots of $p(z) = 0$ lie inside $|z| < 3$.

   (b) (10 points) Find the number of roots of $p(z) = 0$ in the region $1 < |z| < 3$.

3. (20 points) Use complex integration to evaluate

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\, dx.$$

4. (20 points) The series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

converges by the alternating series test. Describe a way to rearrange (or reorder) the terms of the series so that the rearranged series diverges to $+\infty$.

5. Suppose $f(x)$ is continuous for $x \in (0,1)$ and its derivative $f'(x)$ exists for each $x \in (0,1)$.

   (a) (10 points) Suppose $|f'(x)| < B$ for B finite and $x \in (0,1)$. For any $\epsilon > 0$, prove that there exists a $\delta > 0$ such that if

$$0 < a_1 < b_1 < \cdots < a_n < b_n < 1$$

   and $\sum_{j=1}^n |b_j - a_j| < \delta$ then we must have

$$\sum_{j=1}^n |f(b_j) - f(a_j)| < \epsilon.$$

   (b) (10 points) Give an example for which $f(x)$ and $f'(x)$ are both continuous for $x \in (0,1)$ but the ϵ-δ statement in part (a) is false. An informal explanation of why the example works would suffice.