AIM Qualifying Review Exam: Probability and Discrete Mathematics

January 3, 2022

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Suppose that a biased coin that lands on heads with probability p is flipped 10 times (independently). Given that a total of 6 heads results, what is the probability that the first three flips are heads, heads, tails?

Solution

Ross 4.50, p. 177.

We want $P(B|A) = \frac{P(B \wedge A)}{P(A)}$, provided the denominator is non-zero (as it is in this case).

The probability of heads-heads-tails and a total of six heads means four more heads of the seven remaining flips. So

$$\frac{p^2(1-p)\binom{7}{4}p^4(1-p)^3}{\binom{10}{6}p^6(1-p)^4} = \frac{\binom{7}{4}}{\binom{10}{6}}$$

Mathematical concepts: discrete random variables, conditional probability

Problem 2

Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y}, & 0 < x, y \\ 0 & \text{otherwise.} \end{cases}$$

Find $P[X > 1|Y = y]$.

Solution

Ross, pp. 267, Example 5b

The conditional density is

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{e^{-x/y}e^{-y}/y}{e^{-y}\int_0^\infty (1/y)e^{-x/y}dx} \\
&= \frac{1}{y}e^{-x/y}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
P[X>1|Y=y] &= \int_1^\infty \frac{1}{y}e^{-x/y}dx \\
&= -e^{-x/y}\Big|_{x=1}^{\infty} \\
&= e^{-1/y}.
\end{aligned}
$$

Problem 3 A Grey Code of order n is a list of all binary strings of length n listed in an order so that cyclically consecutive strings differ in exactly one bit.

(a) Give a Grey code of order 3.

(b) Describe how to form a Grey code of any order.

Solution

Recurse. Prepend a 0 to each string in a Grey code of order $n-1$, then prepend a 1 to each string in that subcase of order $n-1$, but taken in reverse order.

For example, for $n=3$:

0	0	0
0	0	1
0	1	1
0	1	0
1	1	0
1	1	1
1	0	1
1	0	0

Mathematical concepts: Grey code, recursion

Problem 4

There is available an unlimited number of pennies, nickels, dimes, quarters, and half-dollar pieces, worth 1, 5, 10, 25, and 50 cents each. Let h_n denote the number of ways to form n cents. Determine the generating function $g(x) = h_0 + h_1x + h_2x^2 + \cdots$ in closed form.

Solution

Brualdi, p.221, example.

The generating function for pennies only is $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. For nickels only, we have $1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5}$. Our generating function is the product of these,

$$
\frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}\frac{1}{1-x^{50}}.
$$

For example, the coefficient of x^5 is 2, corresponding to $5 \cdot 1$¢ and $1 \cdot 5$¢. In the series, we get two contributions from $(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^5)$. We convolve the series of coefficients to count the number of ways to combine the coins as a sum.

Mathematical concepts: generating functions, convolution

Problem 5 A Directed Graph is a set of nodes and set of directed edges (u, v), where u and v are nodes,

and (u, v) is said to go from u to v. A Directed Acyclic Graph (DAG) is a directed graph with no directed cycles, i.e., no sequence $u_1, u_2, u_3 \ldots, u_k$ with all directed edges $(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_i)$ and (u_k, u_1) present.

A topological ordering of a directed graph is a sequence of all nodes $v_1, v_2, v_3, \ldots$ such that if there is a directed edge (v_i, v_j) then $i < j$. That is, all edges point left to right in the the topological ordering.

Show that a directed graph G has a topological ordering iff it is acyclic.

Solution

Kleinburg and Tardos, pp. 101–102.

Suppose there is a directed cycle $u_1, u_2, \ldots, u_k, u_1$ then there can be no topological order: If v_i is the first node in the topological order that appears in the cycle, then $v_i = u_j$ has a predecessor u_{j-1} in the cycle. Contradiction.

Conversely, if G has no cycle, then G has a **source**, a node with no predecessor: start at any node, and follow directed edges backward to predecessor and pre-predecessor. This process must terminate at some v_1, since there is no cycle. Put v_1 first in the topological order. Inductively find a topological order of $G - v_1$ and put that order after v_1.

Mathematical concepts: directed graphs, topological order, partial order, induction