AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2022

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

For all $\mathbf{v} \in \mathbb{R}^3$, let $\mathbf{N}\mathbf{v} = \mathbf{w} \times \mathbf{v}$, where $\mathbf{w}$ is a given vector and $\times$ denotes the cross product. Find a complete set of eigenvectors for $\mathbf{N}$ and the corresponding eigenvalues for the following choices of $\mathbf{w}$:

(a) $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution

(a) $\mathbf{N}\mathbf{w} = \mathbf{0}$, so $\mathbf{w}$ is an eigenvector with eigenvalue 0. We also have $\mathbf{N}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{N}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. So $\mathbf{N}$ is a rotation matrix in the $\mathbf{e}_1$-$\mathbf{e}_3$ plane: It has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ \pm i \end{bmatrix}$ with corresponding eigenvalues $\pm i$.

(b) Again $\mathbf{w}$ is an eigenvector with eigenvalue 0, and $\mathbf{N}$ is a rotation matrix in the plane orthogonal to $\mathbf{w}$. To find the eigenvectors, let $\mathbf{w}_\perp = \begin{bmatrix} a \\ b \\ -a-b \end{bmatrix}$ be an eigenvector in this plane. Then $\mathbf{N}\mathbf{w}_\perp = \begin{bmatrix} -a-2b \\ 2a+b \\ b-a \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda(-a-b) \end{bmatrix}$ for eigenvalue λ. We can take $a = 1$ since an eigenvector can be scaled by an arbitrary constant. Setting the components of the vectors equal we solve for λ and b using $b^2+b+1=0$ and $\lambda=-1-2b$. So the eigenvectors are $\begin{bmatrix} 1 \\ -1/2 \pm i\sqrt{3}/2 \\ -1/2 \mp i\sqrt{3}/2 \end{bmatrix}$ with eigenvalues $\mp i\sqrt{3}$.

Problem 2

Let $\mathbf{P} \in \mathbb{R}^{N\times N}$ satisfy $\mathbf{P}^2 = \mathbf{P}$. Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \in \mathbb{R}^N$ be nonzero vectors with the properties that $\mathbf{Pa} = \mathbf{c}$, $\mathbf{Pb} = \mathbf{0}$, and $\mathbf{b}^T\mathbf{c} > 0$. Show that $\|\mathbf{P}\|_2 > 1$, where $\|\mathbf{P}\|_2$ is the matrix 2-norm of $\mathbf{P}$.

Solution

First, some intuition (not required as part of the answer): Since $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}$ is a projection matrix. If $\|\mathbf{P}\|_2 > 1$ it must be an oblique projection—the null space and range are not orthogonal. We use this fact to construct a vector that has components in the null space and range and is lengthened under multiplication by $\mathbf{P}$.

To show $\|\mathbf{P}\|_2 > 1$, we have to show that multiplication by $\mathbf{P}$ increases the length of some vector. Let α be a constant, and consider $\mathbf{P}(\mathbf{c} + \alpha\mathbf{b}) = \mathbf{c}$. We have $\|\mathbf{c} + \alpha\mathbf{b}\|^2 = \|\mathbf{c}\|^2 + 2\alpha\mathbf{c}\cdot\mathbf{b} + \alpha^2\|\mathbf{b}\|^2 < \|\mathbf{c}\|^2$ if $0 > \alpha > -2\mathbf{c}\cdot\mathbf{b}/\|\mathbf{b}\|^2$, so $\mathbf{c} + \alpha\mathbf{b}$ is lengthened by $\mathbf{P}$ for such α.

Problem 3

Find the general solution of the system of **second-order** differential equations

$$\mathbf{y}'' = \mathbf{Ay} + \mathbf{b}\cos t \quad \text{with } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Hint: If you get stuck, first try solving it with $\mathbf{y}'$ in place of $\mathbf{y}''$.

Solution

The general solution is a sum of the homogeneous solution and a particular solution. For the homogeneous solution, plug in $\mathbf{y} = \mathbf{v}e^{rt}$ and get $(r^2\mathbf{I} - \mathbf{A})\mathbf{v} = 0$. The solutions are $r = \pm1, \pm2$ with $\mathbf{v} = \mathbf{e}_1$ for the first pair and $\mathbf{e}_2$ for the second pair. So the homogeneous solution is $\mathbf{y}_c = c_1\mathbf{e}_1e^{-t} + c_2\mathbf{e}_1e^{t} + c_3\mathbf{e}_2e^{-2t} + c_4\mathbf{e}_2e^{2t}$. For the particular solution, try $\mathbf{y}_p = \mathbf{c}\cos t$ (one can also add a $\sin t$ term but it drops out). We get $(-\mathbf{I} - \mathbf{A})\mathbf{c} = \mathbf{b}$, so $\mathbf{c} = \begin{bmatrix} -3/2 \\ -1/6 \end{bmatrix}$.

Problem 4

For the following differential equations, list the sets of initial conditions for which we are guaranteed existence of a unique solution. If possible, state the minimum time for which the solution is guaranteed to exist (as a function of the initial time t_0). Justify your answers.

(a) $(\tan t)\dfrac{d^2z}{dt^2} = z + \log|5t - t^3|$

(b) $(\tan t)\dfrac{d^2z}{dt^2} = \sqrt{|\sin z|}$

Solution

(a) Define $y = z'$ and write it as the linear system $\begin{bmatrix} z \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \cot t & 0 \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \cot t\sqrt{|5t - t^3|} \end{bmatrix}$. By the theorem in chap. 7 of Boyce and DiPrima, we have existence and uniqueness for t_0 where the matrix entries and the second vector on the right hand side are continuous. This holds for $t_0 \neq n\pi$ for integers n and $t_0 \neq \pm\sqrt{5}$. The minimum time of existence is the minimum distance from t_0 to the set $\{\pm5, n\pi\}$.

(b) Again, define $y = z'$ and write it as the nonlinear system $\begin{bmatrix} z \\ y \end{bmatrix}' = \begin{bmatrix} y \\ \cot t\sqrt{|\sin z|} \end{bmatrix}$.

We have local existence and uniqueness for initial conditions (t_0, z_0, y_0) where the right side of the ODE and the Jacobian entries are continuous (Theorem 7.1.1. in Boyce and DiPrima). The Jacobian matrix for the right hand side is $\begin{bmatrix} 0 & 1 \\ (\cot t)(\cos z)\mathrm{sgn}(\sin z)/2\sqrt{|\sin z|} & 0 \end{bmatrix}$. Hence we have local existence and uniqueness for $t_0 \neq n\pi$, $z_0 \neq n\pi$.

Problem 5

Solve Laplace's equation $\nabla^2 u = 0$ in the region $x^2 + y^2 > 1$ (the region outside the unit circle) with the boundary conditions:

$$\hat{n} \cdot \nabla u = 1 \text{ for } x^2 + y^2 = 1$$
$$\lim_{x^2+y^2\to\infty} \frac{u}{xy} = 1.$$

Here $\hat{n}$ is the unit normal vector on the unit circle, pointing outward. Hint: the solution is not unique, so give the most general solution, containing undetermined constants. Also, the Laplacian in rectangular and polar coordinates is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Solution

Using separation of variables, $u(r, \theta) = R(r)Q(\theta)$, we obtain $r^2R''/R + rR'/R = -Q''/Q = k$ for some constants k. Nontrivial periodic solutions Q are $\sin n\theta$ and $\cos n\theta$ for $k = n^2$, n integer. The corresponding solutions for R are found by the ansatz r^α: $r^{\pm n}$ for $n \neq 0$, and the special solutions 1 and $\log r$ for $n = 0$. Hence the general solution is

$$u = c + d\log r + \sum_{n=1}^{\infty} \left(a_n r^{-n} + b_n r^n\right)\sin n\theta + \left(c_n r^{-n} + d_n r^n\right)\cos n\theta.$$

The boundary condition $\partial_r u = 1$ on the unit circle gives $d = 1$; $-a_n + b_n = 0$; $-c_n + d_n = 0$, for $n \neq 0$. The boundary condition at infinity can be written $\lim_{r\to\infty} u/r^2 \sin 2\theta = 1/2$. So $a_2 = b_2 = 1/2$ and the remaining a_n, b_n, c_n, d_n with $n \geq 2$ are zero. Two of the coefficients a_1, b_1, c_1, d_1 may be taken as free parameters, as well as c, so the general solution can be written

$$u = c + \log r + a_1\left(r^{-1} + r\right)\sin\theta + c_1\left(r^{-1} + r\right)\cos\theta + \frac{1}{2}\left(r^{-2} + r^2\right)\sin 2\theta.$$

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