# AIM Qualifying Review Exam in Differential Equations & Linear Algebra

*January 2022*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

**Problem 1**

For all $\mathbf{v} \in \mathbb{R}^3$, let $\mathbf{N}\mathbf{v} = \mathbf{w} \times \mathbf{v}$, where $\mathbf{N}$ is a 3-by-3 matrix, $\mathbf{w}$ is a given vector, and $\times$ denotes the cross product. Find a complete set of eigenvectors for $\mathbf{N}$ and the corresponding eigenvalues for the following choices of $\mathbf{w}$:

(a) $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

**Problem 1**

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**Problem 2**

Let $\mathbf{P} \in \mathbb{R}^{N \times N}$ satisfy $\mathbf{P}^2 = \mathbf{P}$. Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \in \mathbb{R}^N$ be nonzero vectors with the properties that $\mathbf{Pa} = \mathbf{c}$, $\mathbf{Pb} = \mathbf{0}$, and $\mathbf{b}^T\mathbf{c} > 0$. Show that $\|\mathbf{P}\|_2 > 1$, where $\|\mathbf{P}\|_2$ is the matrix 2-norm of $\mathbf{P}$.

**Problem 2**

**Problem 2**

**Problem 2**

**Problem 3**

Find the general solution of the system of **second-order** differential equations

$$\frac{d^2\mathbf{y}}{dt^2} = \mathbf{A}\mathbf{y} + \mathbf{b}\cos t \quad \text{with } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Hint: If you get stuck, first try solving it with $\mathbf{y}'$ in place of $\mathbf{y}''$.

**Problem 3**

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**Problem 4**

For the following differential equations, list the sets of initial conditions for which we are guaranteed existence of a unique solution. If possible, state the minimum length of time for which the solution is guaranteed to exist (as a function of the initial time $t_0$). Justify your answers.

(a) $(\tan t)\dfrac{d^2z}{dt^2} = z + \log|5t - t^3|$

(b) $(\tan t)\dfrac{d^2z}{dt^2} = \sqrt{|\sin z|}$

**Problem 4**

**Problem 4**

**Problem 4**

**Problem 5**

Solve Laplace's equation $\nabla^2 u = 0$ in the region $x^2 + y^2 > 1$ (the region outside the unit circle) with the boundary conditions:

$$\hat{n} \cdot \nabla u = 1 \text{ for } x^2 + y^2 = 1$$
$$\lim_{x^2+y^2 \to \infty} \frac{u}{xy} = 1.$$

Here $\hat{n}$ is the unit normal vector on the unit circle, pointing outward. Note: the solution is not unique, so give the most general solution, containing undetermined constants. Also, the Laplacian in rectangular and polar coordinates is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

**Problem 5**

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