AIM Qualifying Exam: Advanced Calculus and Complex Variables

January 2022

There are five (5) problems in this examination, each worth 20 points.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. Suppose $f : [0,1] \to \mathbb{R}$ is a function that is strictly increasing and continuous with $f(0) < f(1)$. Suppose that

$$f(0) = y_0 < y_1 < \cdots < y_n = f(1)$$

is a division of the interval $[f(0), f(1)]$.

   a) (12 points) For $I_j = [y_j, y_{j+1}]$, $j = 0, \ldots, n-1$, prove that $f^{-1}(I_j)$, or $f^{-1}[y_j, y_{j+1}]$, is a closed interval. Denote the length of $f^{-1}[y_j, y_{j+1}]$ by μ_j and let

$$S = \sum_{j=0}^{n-1} y_j \mu_j.$$

   If $\delta = \max_{j=0,\ldots,n-1} |y_j - y_{j+1}|$, does S converge as $\delta \to 0$ and if so to what quantity? Justify your answer.

   b) (8 points) Suppose now that $f(0) < f(1)$ and that f is increasing but not necessarily strictly increasing. Does S converge as $\delta \to 0$ and if so to what quantity?

2. (20 points) Suppose $f : [0, 1] \to \infty$ is an increasing function but not necessarily continuous. Let S denote the set of points $x \in [0, 1]$ at which f is discontinuous. Prove that S is a countable set.

3. Let $f(z) = \arctan z$ with $f(0) = 0$.

a) (5 points) Obtain the Taylor series of $f(z)$ in the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

and determine its radius of convergence.

b) (15 points) Suppose $f(z)$ is expanded in a Taylor series of the form

$$f(z) = a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots$$

with $a_0 = f(1) = \frac{\pi}{4}$. What is its radius of convergence?

4. (20 points) Let $f(z)$ be a complex valued function that is continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Assume that $|f(z)| = 1$ if $|z| = 1$. Suppose the definition of f is extended to $|z| > 1$ using

$$f(z) = \frac{1}{\bar{f}\left(\frac{1}{\bar{z}}\right)},$$

where $\bar{z}$ is the complex conjugate.

a) Show that $f(z)$ has a complex derivative for $|z| > 1$ and calculate this derivative in terms of f' and f evaluated at $|z| < 1$.

b) Show that $f(z)$ is continuous for $|z| = 1$.

5. (20 points) Evaluate

$$\int_0^\infty \frac{dx}{x^{\frac{1}{4}}(1+x)}.$$