# AIM Qualifying Review Exam in Differential Equations & Linear Algebra

*August 2022*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

**Problem 1**

Consider a real matrix $\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}$ where $\mathbf{A}$ is a symmetric square matrix, $\mathbf{B}$ is not necessarily square, and $\mathbf{B}^T$ is the transpose of $\mathbf{B}$.

(a) (8 points) Show that $\mathbf{C}$ is singular if the number of columns of $\mathbf{B}$ is strictly larger than the number of its rows.

(b) (12 points) Show that if $\mathbf{A}$ is strictly positive definite, then $\mathbf{C}$ is nonsingular if and only if the columns of $\mathbf{B}$ are linearly independent.

**Problem 1**

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**Problem 2**

Let $\mathbf{A}$ be a 2-by-2 matrix with complex entries that is Hermitian: $\mathbf{A} = \mathbf{A}^*$. Let $\mathbf{a}$ be a column vector, and let $\mathbf{B} = \mathbf{A} + \mathbf{a}\mathbf{a}^*$. Denote the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ by $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ respectively.

(a) (5 points) Show that $\mathbf{a}\mathbf{a}^*$ is a 2-by-2 matrix that is positive semi-definite. Explain why the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are real.

(b) (10 points) Assume that the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are ordered so that $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Show that $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2$.

(c) (5 points) Find the eigenvalues and eigenvectors of $\mathbf{B}$ if $\alpha_1 = \alpha_2$.

**Problem 2**

**Problem 2**

**Problem 2**

**Problem 3**

(a) (10 points) Find the first three nonzero terms in each independent series solution about $x = 0$ for the following differential equation

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y = 0.$$

(b) (10 points) Find three independent **real** solutions to the equation

$$x^3\frac{d^3y}{dx^3} + 6y = 0.$$

valid in the domain $x > 0$.

**Problem 3**

**Problem 3**

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**Problem 4**

Find the general solution to the system of differential equations

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2+y^2}}(x^2+y^2-4),$$
$$\frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2+y^2}}(x^2+y^2-4).$$

Describe the qualitative behavior of solutions for all initial conditions $(x,y)|_{t=0} = (x_0,y_0) \in \mathbb{R}^2$.

**Problem 4**

**Problem 4**

**Problem 4**

**Problem 5**

Solve the PDE

$$\partial_{xx}u + 2\partial_x u + \partial_{yy}u = 0$$

for $u(x, y)$ on the square domain $0 \leq x, y \leq 1$ with the boundary conditions:

$$u(x, 0) = 0 \; ; \; u(x, 1) = 1 \; ;$$
$$u(0, y) = 0 \; ; \; u(1, y) = 1.$$

**Problem 5**

**Problem 5**

**Problem 5**