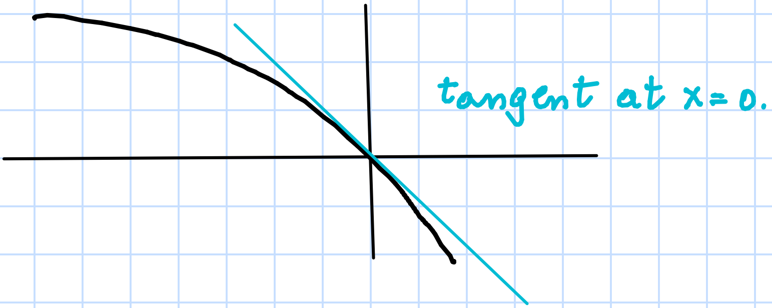
1. $\left|\frac{a_{n+1}}{a_n}\right| = 1 - \frac{\alpha}{n+1}$

$\Rightarrow \log|a_{n+1}| - \log|a_n| = \log\left(1 - \frac{\alpha}{n+1}\right)$

$\log(1-x) \leq -x$ for $-\infty < x < 1$

which may be proved from the picture



Thus $\log|a_{n+1}| - \log|a_n| \leq -\frac{\alpha}{n+1}$

or $\log|a_{n+1}| \leq \log|a_n| - \frac{\alpha}{n+1}$

$\leq \log|a_{n-1}| - \alpha\left(\frac{1}{n} + \frac{1}{n+1}\right)$

$\vdots$

$\leq \log|a_{n_0}| - \alpha\left(\frac{1}{n_0+1} + \cdots + \frac{1}{n+1}\right)$

with $n_0 = \max(\lceil \alpha - 1 \rceil, 0)$ and $n > n_0$.

Next,

$$\frac{1}{n_0+1} + \cdots + \frac{1}{n+1} \geq \int_{n_0+1}^{n+2} \frac{dx}{x}$$ (using left Riemann sum)

$$= \log \frac{n+2}{n_0+1}.$$

Therefore

$$\log |a_{n+1}| \leq \log |a_{n_0}| - \alpha \log \frac{n+2}{n_0+1}$$

or

$$|a_{n+1}| \leq |a_{n_0}| \cdot \left(\frac{n_0+1}{n+2}\right)^{\alpha}$$

For $n > n_0$.

We may choose a constant C large enough such that

$$|a_{n+1}| \leq C\,(n+2)^{\alpha}$$

or

$$|a_n| \leq C\,(n+1)^{\alpha} \quad \text{for } n = 0, 1, 2, \ldots$$

Binomial Series of $(1+z)^{1/2}$

$$(1+z)^{1/2} = 1 + \frac{1}{2} z + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!} z^2 + \cdots$$

$$= 1 + \frac{1}{2} z - \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{2!} z^2 + \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{3!} z^3 - \cdots$$

The coeff of z^n is

$$\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}$$

$$= \frac{(-1)^{n-1}}{2^n \cdot n!} \cdot 1\cdot 3\cdot 5 \cdots (2n-3)$$

Thus $\left|\frac{a_{n+1}}{a_n}\right| = |z| \frac{(2n-1)}{2(n+1)} = \frac{2n+2-3}{2(n+1)} \cdot |z|$

$$= \left|1 - \frac{3/2}{n+1}\right| |z|.$$

If $|z|=1$ then $|a_n| \leq C(n+1)^{-3/2}$ for $n = 0, 1, 2 \ldots$

Because $\sum n^{-3/2} < \infty$ and by the M-test, the binomial series converges uniformly for $|z| \leq 1$.

2. The sum of the series is AB.

$$(a_0 + \cdots + a_n)(b_0 + \cdots + b_n)$$
$$=$$
$$\begin{aligned} & a_0 b_0 \\ & + (a_0 b_1 + a_1 b_0) \\ & + (a_0 b_2 + a_1 b_1 + a_2 b_0) \\ & + \quad \vdots \\ & + (a_0 b_n + \cdots + a_n b_0) \\ & + (a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) \\ & + \quad \vdots \\ & + a_n b_n \end{aligned}$$

Therefore $(a_0 + \cdots + a_n)(b_0 + \cdots + b_n) \geq c_0 + \cdots + c_n$.

Because $a_j > 0$ and $b_j > 0$, we must have

$$AB \geq c_0 + \cdots + c_n$$

Thus the series $c_0 + c_1 + \cdots$ converges and $AB \geq c_0 + c_1 + \cdots$

For the other direction, choose n so large that

$$a_{n+1} + a_{n+2} + \cdots < \epsilon$$

and

$$b_{n+1} + b_{n+2} + \cdots < \epsilon.$$

Then

$$AB = (a_0 + \cdots + a_n + a_{n+1} + \cdots)$$
$$(b_0 + \cdots + b_n + b_{n+1} + \cdots)$$

$$\leq (a_0 + \cdots + a_n)(b_0 + \cdots + b_n)$$
$$+ (a_{n+1} + \cdots)(b_0 + b_1 + \cdots)$$
$$+ (b_{n+1} + \cdots)(a_0 + a_1 + \cdots)$$

$$= (a_0 + \cdots + a_n)(b_0 + \cdots b_n) + A\epsilon + B\epsilon$$

$$\leq c_0 + c_1 + \cdots + c_{2n} + A\epsilon + B\epsilon$$

The limit $n \to \infty$ with arbit small ϵ shows that

$$c_0 + c_1 + c_2 + \cdots = AB.$$

3. By the maximum modulus principle

$$|f(z)| \le 1$$

for $|z| \le 1$.

Suppose $f(a) \neq 0$ for any a with $|a| < 1$. Then

$$\frac{1}{f(z)}$$

is analytic in a nghd of $|z| \le 1$.

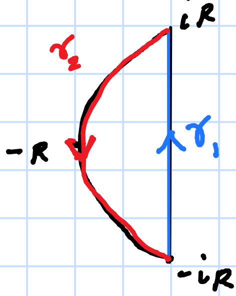
Again applying the max modulus thm we get

$$\frac{1}{|f(z)|} \le 1$$

for $|z| \le 1$, or $|f(z)| = 1$ for $|z| \le 1$.

4. $$\int_{-i\infty}^{i\infty} \frac{e^{z+1}}{z+1}\, dz .$$

Consider γ :



$$\int_\gamma \frac{e^{z+1}}{z+1}\, dz = 2\pi i \operatorname{Res}\left(\frac{e^{z+1}}{z+1} ; z = -1\right) = 2\pi i .$$

Thus $2\pi i = \displaystyle\int_{\gamma_1} \frac{e^{z+1}}{z+1}\,dz + \int_{\gamma_2} \frac{e^{z+1}}{z+1}\,dz.$

$$\lim_{R\to\infty} \int_{\gamma_1} \frac{e^{z+1}}{z+1}\,dz = \int_{-i\infty}^{i\infty} \frac{e^{z+1}}{z+1}\cdot dz.$$

$$\left| \int_{\gamma_2} \frac{e^{z+1}}{z+1}\cdot dz \right| \le \int_{\gamma_2} \frac{|e^{z+1}|}{|z+1|}\,|dz|$$

$$\le \frac{1}{R-1} \int_{\gamma_2} |e^{z}|\,dz$$

$$= \frac{1}{R-1} \int_{\pi/2}^{3\pi/2} e^{R\cos\theta} \cdot R\,d\theta$$

If $\theta = \frac{\pi}{2} + u$

$$\left| \int_{\gamma_2} \frac{e^{z+1}}{z+1}\,dz \right| \le \frac{1}{R-1} \int_0^{\pi} e^{-R\sin\theta} \cdot R\,d\theta$$

$$\le \frac{1}{R-1} \int_0^{\pi} e^{-R\cdot\frac{2\theta}{\pi}} \cdot R\cdot d\theta$$

$$= 0$$

Thus $\displaystyle\int_{-i\infty}^{i\infty} \frac{e^{z+1}}{z+1}\cdot dz = 2\pi i.$

5. By the argument principle

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = Z - P$$

where Z is the number of zeros of f in $|z|<1$ and P is the number of poles.

f obviously has a double pole at $z=0$ but no other. Thus $P=2$.

Because $\int_\gamma \frac{f'}{f} = 0$ we also have $Z=2$.

The roots of the cubic eqn in $|z|<1$ satisfy

$$z^2 f(z) = 0$$

Because $d \neq 0$, $z=0$ is not a root.

Thus the roots satisfy $f(z)=0$.

The cubic equation has 2 roots in $|z|<1$.