AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2021

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Find a real matrix $\mathbf{Q}$ so that for any $\mathbf{v} \in \mathbb{R}^3$, $\|\mathbf{Q}\mathbf{v}\|_2$ is the distance from $\mathbf{v}$ to the set of points $\{x, y, z\}$ that satisfy both $x - y + 2z = 0$ and $x + 3z = 0$.

(b) The solution for $\mathbf{Q}$ in part a is not unique. What are the possible eigenvalues of any member of the set of solutions?

Problem 1

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Problem 2

(a) Let $\mathbf{A}$ be a 2-by-2 matrix with trace (sum of diagonal entries) equal to 5 and determinant equal to 6. What is the characteristic polynomial of $\mathbf{A}^2$?

(b) Does $\mathbf{I} + \mathbf{A}^{-2} + \mathbf{A}^{-4} + \mathbf{A}^{-6} + \ldots$ converge? If so, what are the eigenvalues of the matrix it converges to?

(c) Let $\mathbf{B}$ be a 2-by-2 matrix that has the same trace and determinant as $\mathbf{A}$. Are $\mathbf{B}$ and $\mathbf{A}$ similar matrices? Explain.

Problem 2

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Problem 2

Problem 3

(a) Find the general solution of the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{9}y = 0$$

(b) Find the first two terms in two independent series solutions, expanded about $x = 0$, for the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x - \frac{1}{9}\right) y = 0.$$

Problem 3

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Problem 4

(a) Find the general solution to the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}. \tag{1}$$

and sketch the phase portrait. Include at least five trajectories in total, with a typical trajectory in each region of the plane. Put arrows on the trajectories showing the direction of motion in forward time. Classify the type of critical point (e.g. spiral sink, source, etc.) and its stability.

(b) Find a solution to the system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -e^{-2t} \\ -3e^{-2t} \end{bmatrix} \tag{2}$$

Problem 4

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Problem 5

(a) Solve Laplace's equation

$$\partial_{rr}u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_{\theta\theta}u = 0$$

for $u(r,\theta)$ in a half-disk $\{0 \leq r < 1, 0 < \theta < \pi\}$ with the boundary conditions:

$$u(r,0) = 0\ , 0 < r \leq 1,$$
$$u(r,\pi) = \pi\ ,\ 0 \leq r \leq 1,$$
$$u(1,\theta) = \theta\ \ ,\ 0 \leq \theta \leq \pi.$$

(b) Solve the same equation as in part a but now for $u(r,\theta)$ in a half-annulus $\{1 < r < 2, 0 < \theta < \pi\}$ with the boundary conditions:

$$\partial_\theta u(r,0) = \partial_\theta u(r,\pi) = 0\ \ ,\ 0 \leq r \leq 1,$$
$$u(1,\theta) = 1\ \ ,\ u(2,\theta) = \cos 2\theta\ \ ,\ 0 \leq \theta \leq \pi.$$

Problem 5

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