AIM Qualifying Exam: Advanced Calculus and Complex Variables
January 2021
For full credit, support your answers with appropriate explanations.
There are five problems, each worth 20 points.

1. (20 points) Suppose $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$, and $f : [0,1] \to \mathbb{R}$ are continuous functions. Suppose further that $f_n(x) \to f(x)$ uniformly for $x \in [0,1]$. Given $x_0 \in (0,1)$ and $\epsilon > 0$, prove that there exists $\delta > 0$ such that

$$|f_n(x) - f_n(x_0)| < \epsilon$$

for $n = 1, 2, \ldots$ if $|x - x_0| < \delta$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the second derivative f'' is continuous. Assume $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

   (a) (10 points) If $x < y < z$, prove that

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(y) - f(z)}{y - z}.$$

   (b) (10 points) If $x < y$ and $\alpha \in [0,1]$, prove that

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y).$$

3. (20 points) Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function. Let B be the least upper bound of the quantities

$$|f(x_0) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_{n-1}) - f(x_n)|$$

with

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$$

and x_j otherwise arbitrary and n allowed to be any positive integer. Derive a formula for B in terms of f.

4. (20 points) Use complex integration to evaluate

$$\int_{-\infty}^{+\infty} \frac{\left(e^{i\alpha x} - 1\right)\left(e^{i\beta x} - 1\right)}{x^2}\, dx,$$

assuming $0 < \alpha < \beta$.

5. The problem has two parts.

   (a) (10 points) If $f(z)$ is analytic for $z \in \mathbb{C}$ (entire function) and satisfies the bound $|f(z)| \leq A|z| + B$ for some $A, B > 0$, prove that $f(z)$ must be of the form $az + b$.

   (b) (10 points) Determine the form of $f(z)$ if $f(z)$ is analytic for all $z \in \mathbb{C}$ except $z = 0$, the singularity at $z = 0$ is a simple pole, and $f(z)$ satisfies $|f(z)| < A|z| + B$ if $|z| > C$, where A, B, C are positive constants.