

# AIM Preliminary Exam: Probability & Discrete Mathematics

*January 7, 2014*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

**Problem 1**

Let  $A$  be a subset of size 101 from the set  $\{1, 2, 3, \dots, 200\}$  (of size 200). Show that  $A$  contains an  $x$  and a  $y$  such that  $x$  divides  $y$ .

Problem 1

Problem 1

Problem 1

**Problem 2**

Determine the number of permutations of  $\{1, 2, \dots, 6\}$  in which no integer is in its natural position.

Give an actual number and explain your reasoning. Note:  $6! = 720$ .

Problem 2

Problem 2



Problem 2

### **Problem 3**

Consider Euclid's algorithm for finding the greatest common divisor of two non-negative inputs,  $x$  and  $y$ :

```
Euclid( $x, y$ ):  
  if ( $x > y$ ) swap  $x$  and  $y$   
  if  $x = 0$  output  $y$   
  output Euclid( $y \bmod x, x$ )
```

You may assume existence and uniqueness of prime factorization and the implication that, if  $c$  divides  $x$  and  $y$ , then  $c$  divides the gcd of  $x$  and  $y$ , denoted  $(x, y)$ .

- (a) Show that the algorithm's output  $z$  is a common divisor of  $x$  and  $y$ , by stating and proving an appropriate *loop invariant*, that holds at the entrance to the loop/recursive call, holds at iteration  $k + 1$  if it holds at iteration  $k$ , and is useful at the loop's termination. (This step and the next can potentially be combined into a single argument.)
- (b) Show that the algorithm's output  $z$  is the *greatest* common divisor of  $x$  and  $y$ , by showing that, if  $w$  is any common divisor, then  $w$  divides  $z$ . Again, use a loop invariant.
- (c) Show that the algorithm takes  $O(\log n)$  iterations if  $x$  and  $y$  are both at most  $n$ .

Problem 3

Problem 3

Problem 3

**Problem 4**

Let  $X$  be a Poisson random variable with parameter  $\lambda$ . What value of  $\lambda$ , in terms of  $k \geq 0$ , maximizes  $\Pr(X = k)$ ?

Problem 4

Problem 4



Problem 4

**Problem 5**

A certain blood test may result detect bacteria of type A, bacteria of type B, or no bacteria (but not both). Bacteria of type A is detected with probability  $p$ ; bacteria of type B is detected with probability  $q$ ; and no bacteria with probability  $1 - p - q$ . Suppose that a series of independent blood tests are performed continually until some bacteria is found. Compute the probability that bacteria of type A is detected before bacteria of type B.

Problem 5

Problem 5

Problem 5