AIM Preliminary Exam: Probability & Discrete Mathematics

January 8, 2013

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

For each of the following statments, decide whether it is true or false. If it is true, give a proof. If it is false, give a counterexample:

(a) Let G be a connected, undirected graph with a distinct cost $c(e)$ on each edge e. Suppose e^* is the cheapest edge in G. Then there is a minimum spanning tree T of G that does *not* contain e^*.

(b) Let G be a connected, undirected graph with a distinct *positive* cost $c(e)$ on each edge e and let T be a minimum spanning tree. True or false: T is also a minimum spanning tree for the squared cost $c^2(e)$.

Problem 1

Problem 1

Problem 1

Problem 2

We are given an arrangement of n lines in the plane, $L_1, L_2, \ldots, L_n$, where we assume that no line is vertical and no three lines intersect at a point. Denote by *upper hull* the pointwise maximum of all the lines—a polygonal curve, given by coordinates of its vertices, sorted by horizontal coordinate. (Finding the upper hull is related to finding the visible parts of a complex scene.)

(a) Give an algorithm that takes a collection of n lines and, in time $O(n \log n)$, returns the upper hull. *Hint: Show how to find the upper hull of a set of $n = 1$ line(s) and show how, given the upper hull H' of $\{L_1, L_2, \ldots, L_{n/2}\}$ and the upper hull H'' of $\{L_{n/2+1}, L_{n/2+2}, \ldots, L_n\}$, to construct the upper hull H of $\{L_1, L_2, \ldots, L_n\}$ from H' and H''.*

(b) Reduce the problem of sorting integers to this problem, by a linear-time mapping. (Since sorting requires time $\Omega(n \log n)$ in many models, we conclude that the above algorithm is tight.)

Problem 2

Problem 2

Problem 2

Problem 3

Let X and Y be independent exponential random variables, *i.e.*, with density e^{-x} for $x \geq 0$ and 0 for $x < 0$. Express the density function of

(a) $Z = X/Y$

(b) $V = \min(X, Y)$

as integrals in terms of the density functions of X and Y and evaluate.

Problem 3

Problem 3

Problem 3

Problem 4

If X and Y are independent binomial random variables with identical parameters n and $p = 1/2$, show analytically that the conditional distribution of X given that $X + Y = m$ is the hypergeometric distribution. Also, give a combinatorial argument (without computation). *Hint: Suppose that* $2n$ *coins are flipped. Let* X *denote the number of heads in the first* n *flips and* Y *denote the number of heads in the second* n *flips. Argue that given a total of* m *heads, the number of heads in the first* n *flips has the same distribution as the number of white balls selected when a sample of size* m *is chosen from* n *white and* n *black balls.*

Problem 4

Problem 4

Problem 4

Problem 5

(a) How many distinct positive divisors does $11^2 \cdot 13^3 \cdot 17^5 \cdot 19^7$ have?

(b) What is the largest power of 10 that divides $n!$, *i.e.*, $n(n-1)(n-2)\cdots 2 \cdot 1$?

(c) Evaluate

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \cdots + \frac{1}{n+1}\binom{n}{n}.$$

Problem 5

Problem 5

Problem 5