AIM Preliminary Exam: Probability & Discrete Mathematics

January 9, 2011

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) Find the sum of all four-digit numbers that can be obtained by using (without repetition) the digits 2, 3, 5, and 7.

(b) Suppose that $\leq$ denotes a partial ordering on a set S. Prove that if $n = 2, 3, 4, \ldots$, there cannot exist distinct elements $s_1, s_2, \ldots, s_n \in S$ such that $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_1$.

Problem 1

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Problem 2

Let D_n denote the number of diagonals of an n-sided convex polygon in the plane. For example, $D_2 = D_3 = 0$, $D_4 = 2$, and $D_5 = 5$. Find D_n for all $n \geq 3$, and prove the validity of your formula.

Problem 2

Problem 2

Problem 2

Problem 3

Let X be an exponential random variable with parameter $\lambda = 1$, and let Y_i, $i = 1, 2, 3, \ldots$ be exponential random variables with parameter $\lambda = 2$. Assume that X and all of the Y_i are independent.

(a) Let

$$S_n := \frac{1}{n} \sum_{i=1}^{n} Y_i, \quad n = 1, 2, 3, \ldots.$$

Recall an appropriate theorem to deduce the limit $\lim_{n \to \infty} \mathbb{P}(X > S_n)$.

(b) Compute the exact value of $\mathbb{P}(X > S_n)$ as a function of n in closed form. Evaluate $\lim_{n \to \infty} \mathbb{P}(X > S_n)$ to directly confirm part (a).

Problem 3

Problem 3

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Problem 4

(a) Let $B = 1, 2, 3, \ldots$ be a discrete random variable with $\mathbb{P}(B = k) = Aw^k$ where $0 < w < 1$. Find A and $\mathbb{E}[B]$.

(b) Let X_j for $j = 1, 2, 3, \ldots$ be independent continuous random variables for each j. Assume that X_j is exponentially distributed with parameter αj. So, for example, X_3 is an exponential random variable with parameter 3α. Find $\mathbb{E}[X_1 + X_2 + X_3]$.

(c) Define a random variable as follows:

$$R = X_j \quad \text{if} \quad B = j \quad \text{for} \qquad j = 1, 2, 3, \ldots$$

where B is the random variable defined in part (a) and X_j are the random variables defined in part (b). Find the probability density function of R. Find $\mathbb{E}[R]$.

Problem 4

Problem 4

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Problem 5

Let $X_1, \ldots, X_m$ be literals that are either TRUE or FALSE. Consider clauses of the form $C = W \vee Y \vee Z$ where each of the variables W, Y, and Z stands for one of the m literals X_i or its negation $\neg X_i$. The 3-SAT problem is to determine whether a given collection of n such clauses admits an assignment of truth values to the m literals making each of the clauses TRUE.

Find a polynomial (in m and n) time reduction from the 3-SAT problem to the k-CLIQUE problem of determining whether a given graph $G = (V, E)$ has a set of k vertices that are all connected to each other by edges, forming a complete subgraph.

Problem 5

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