AIM Preliminary Exam: Probability and Discrete Mathematics

September 5, 2016

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Fix $\lambda > 0$, and let X be a random variable with $\Pr(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}$, for $i = 0, 1, 2, 3, \ldots$, and $\Pr(X = i) = 0$, otherwise.

(a) Show that $\Pr(X = i)$ increases then decreases, as i increases from 0 to ∞. What value(s) of i maximizes $\Pr(X = i)$?

(b) Find $\Pr(X \text{ is even})$ by analyzing the Taylor series for e^{λ} and similar functions.

Problem 1

Problem 1

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Problem 2

(a) Suppose the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x,y > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y independent?

(b) What about

$$g(x,y) = \begin{cases} 2, & 0 < x < y, 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Sketch the region where g is non-zero.

Problem 2

Problem 2

Problem 2

Problem 3 You are given $2n$ distinct numbers in **two** sorted arrays of length n each. You may assume n is a power of 2, or any similar convenient condition. Give an algorithm that finds the median two elements of all $2n$ and such that the algorithm runs in time $O(\log n)$. (The median two elements, of this even-sized universe, are such that $n-1$ elements are smaller and $n-1$ elements are greater. E.g., in $\{1, 2, 3, 4, 5, 6\}$ the median two elements are 3 and 4.) Prove correctness formally using a loop invariant.

Problem 3

Problem 3

Problem 3

Problem 4

Consider a three-dimensional grid whose dimensions are 10 by 15 by 20. How many different paths of 45 unit-length steps are there from the front-lower-left corner to the diagonally opposite corner? (You need not simplify arithmetic.)

Problem 4

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Problem 4

Problem 5

A **Knight's Tour** is a sequence of squares on a chessboard in which consecutive squares are at opposite corners of a 2×3 or 3×2 rectangle, the first square is the same as the last, and each square on the board is listed exactly once (except for the first/last, which is mentioned twice.) An example tour fragment is below.

Prove that there is no Knight's Tour on a 4×4 board.

	2		
4			
1		3	

Problem 5

Problem 5

Problem 5