

AIM Preliminary Exam: Probability and Discrete Mathematics

September 7, 2015

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

H/T Ross, 3e, 6.21

Suppose X is a standard normal random variable, i.e., with density $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. Find the density $\phi(y)$ of $Y = e^X$. Show work.

Problem 1

Problem 1

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Problem 2

(a) H/T Ross, 3e, 3.5.

Let E and F be events, $\Pr(E), \Pr(F) > 0$, and suppose $\Pr(E|F) < \Pr(E)$. Does it follow that $\Pr(F|E) < \Pr(F)$, that $\Pr(F|E) > \Pr(F)$, or neither? Prove or give a counter example.

(b) Suppose an ι fraction of the population has rabies. A sick person causes a positive test outcome with probability $1 - \delta$ and a healthy person causes a positive test outcome with probability ϵ . If a random patient causes a positive test outcome, what is the probability that the patient has rabies? What happens as ι, ϵ, δ go to zero, possibly at different rates?

Problem 2

Problem 2

Problem 2

Problem 3

H/T: Brualdi 5.7.12.

Let n be a positive integer. Prove that

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 = \begin{cases} 0, & n \text{ odd;} \\ (-1)^m \binom{2m}{m}, & n = 2m. \end{cases}$$

Hint: $(1-x^2)^n = (1+x)^n(1-x)^n$. You may take as definition that $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$.

Problem 3

Problem 3

Problem 3

Problem 4

Let $M_n = \mathbf{1}_n - I_n$, the n -by- n matrix consisting of zeros on the main diagonal and ones elsewhere, e.g., :

$$M_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We will find the determinant of M_n , after a digression. For each $j \leq n$, consider the set P_j of permutations on $\{1, 2, 3, \dots, n\}$ fixing each element of $\{1, 2, 3, \dots, j\}$.

- (a) The set D_n of derangements on n numbers is the set of permutations on n numbers such that no number is mapped by the permutation to its original position, i.e., permutations π such that, for all i , $\pi(i) \neq i$. What is $|D_n|$? Consider P_j and use Inclusion-Exclusion to write a summation expression involving things like $j!$, 2^j , and $\binom{n}{j}$.
- (b) Prove that no permutation can be the product of an even number of pairwise swaps and also the product of an odd number. Hint: count *inversions*, i.e., elements i and j with $i < j$ but $\pi(i) > \pi(j)$. (This justifies the *sign* of a permutation π as -1 raised to the power equal to the number of swaps in some representation of π .)
- (c) What fraction of P_j consists of even permutations (sign $+1$)?
- (d) Define the determinant of matrices like M_n by

$$\det(M_n) = \sum_{\pi \in D_n} \text{sign}(\pi),$$

where the sum is over derangements. What is the determinant of M_n ?

Problem 4

Problem 4

Problem 4

Problem 5

We are presented with n matrices to multiply. The matrices are **not** assumed to be square. Our job is to look only at the sizes of the matrices (not the entries) and to determine the best parenthesization of the matrices for the naive matrix multiplication algorithm, where “best” minimizes the number of scalar multiplications.

For example, suppose $A_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A_2 = (3 \ 4 \ 5)$, and $A_3 = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$.

Then $(A_1A_2)A_3$ requires 6 multiplications to form $M = A_1A_2$ (a 2×3 matrix) and another 6 multiplications to form MA_3 , for a total of 12. Better is $A_1(A_2A_3)$, which takes 3 multiplications to form the 1×1 matrix $M' = A_2A_3$ and another 2 to form A_1M' , for a total of 5 multiplications.

Use the following notation: r_i is the number of rows of matrix i and $c_i = r_{i+1}$ is the number of columns of matrix i . So, formally, our input is the r_i 's and c_i 's, not the matrices. It takes $r_1r_2c_2 = r_1c_1c_2$ scalar multiplications to compute A_1A_2 , since each entry in the $r_1 \times c_2$ result requires $c_1 = r_2$ multiplications.

- (a) Show that the number of possible parenthesizations in a chain of n matrices is exponential in n . Note: You need not find the exact number of parenthesizations.
- (b) Give an algorithm that runs in time polynomial in n and that computes, for all $1 \leq i \leq j \leq n$, the number $m(i, j)$ of scalar multiplications needed to compute $M_{ij} = A_iA_{i+1} \cdots A_{j-1}A_j$ optimally. Hint: find an algorithm that fills an $n \times n$ table indexed by (i, j) , describe the order in which the table is filled, and explain what information is needed to determine the i, j cell. Note that M_{ij} is an $r_i \times c_j$ matrix.

Problem 5

Problem 5

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