AIM Preliminary Exam: Probability & Discrete Mathematics

September 1, 2013

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Find a formula for the n'th Fibonacci number f_n, given by

$$\begin{cases} f_n = f_{n-1} + f_{n-2}, \quad n \geq 2 \\ f_1 = 1 \\ f_0 = 0 \end{cases}$$

Problem 1

Problem 1

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Problem 2

Suppose we have a flippable coin of unknown heads probability p and we want to simulate a fair coin. For each of the two experiments described below, give the probability that the experiment outputs heads. Assume all flips of the coin are independent.

(a) Repeatedly flip the coin (twice at a time) until the $2n$'th outcome differs from the $2n-1$'st outcome, for some integer n. Output the outcome of the last flip.

(b) Repeatedly flip the coin until the n'th outcome differs from the $n-1$'st outcome, for some integer n. Output the outcome of the last flip.

Problem 2

Problem 2

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Problem 3 Let X be Poisson distributed with parameter λ, i.e., $\Pr(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, for $k = 0, 1, 2, 3, \ldots$.

(a) Show that $E[X^n] = \lambda E[(X + 1)^{n-1}]$.

(b) Compute $E[X^3]$.

Problem 3

Problem 3

Problem 3

Problem 4 Let (X, Y) be a random point chosen according to the uniform distribution in the disc of radius 1 centered at the origin.

(a) Compute the densities of X and of Y.

(b) Are X and Y independent?

(c) Compute the joint density of the polar coordinates $R = \sqrt{X^2 + Y^2}$ and $\Theta = \tan^{-1}(Y/X)$.

(d) Are R and Θ independent?

Problem 4

Problem 4

Problem 4

Problem 5

Given a finite sequence, like $(9, 4, 7, 5, 8)$, the subsequences $(4, 5)$ and $(4, 8)$ are increasing subsequences. The sequence $(4, 5, 8)$ is a longest increasing subsequence, since there are no increasing subsequences of length 4. (9 cannot be used and at most one of 7 and 5 can be used.)

Give an efficient algorithm for finding the longest increasing subsequence of a given sequence of length n. Give a (useful) bound on the runtime of your algorithm in the form an^b or c^n, where the constant a may be left unspecified, but the constants b or c should be given. You may assume that the given sequence is a permutation of $(1, 2, 3, \ldots, n)$.

Problem 5

Problem 5

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