

AIM Preliminary Exam: Probability & Discrete Mathematics

September 2, 2012

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

A *min-heap* data structure on n items is a tree structure indexing an array $\mathbf{a}[0..n-1]$ of length n . For any position (node) $i > 0$, the *parent* of i is position $\lfloor (i-1)/2 \rfloor$. For any $i \geq 0$, the *left child* is position $2i+1$ and the right child is position $2i+2$. Each data item has a key and the keys of two data items can be compared.

The structure is said to satisfy the *min-heap property* (briefly, *to be a heap*) if, at each node ν with subtree τ , the key at ν is the minimum over keys in τ .

- (a) (Insertion.) Suppose we start with a heap and add a new item $\mathbf{a}[n]$ to the array, temporarily violating the min-heap property. Give an efficient algorithm to restore the min-heap property. Prove correctness and give a worst-case bound on the runtime of your algorithm (up to a constant factor).
- (b) (Deletion.) Suppose we start with a heap and replace $\mathbf{a}[0]$ with an item whose key is $+\infty$ *i.e.*, larger than all other keys in the heap, temporarily violating the heap property. Give an efficient algorithm to restore the min-heap property. Prove correctness and give a worst-case bound on the runtime of your algorithm (up to a constant factor).
- (c) Give an algorithm to sort n numbers in time $O(n \log n)$, accessing the data only through the two operations insertion and deletion above, in addition to related and auxillary operations such as `parent(j)`, `rightchild(j)`, `output_root`, `new_heap` — that creates a new heap for $n=0$ — etc. Do not give a brand new sorting algorithm from scratch.

You may omit some corner cases if they are handled in a straightforward way. Each proof of correctness should state and prove a loop invariant.

Problem 1

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Problem 2

- (a) How many solutions in non-negative integers are there to $w + x + y + z = 30$?
- (b) How many solutions in non-negative integers are there to the following?

$$\begin{aligned}w + x + y + z &= 30 \\ 2 \leq w &\leq 9.\end{aligned}$$

You may use binomial coefficients in your answer.

Problem 2

Problem 2

Problem 2

Problem 3

- (a) Exhibit a graph with $\frac{(n-1)(n-2)}{2}$ edges that is *not* connected.
- (b) Show that a graph on n vertices that has at least $\frac{(n-1)(n-2)}{2} + 1$ edges *is* connected.
- (c) Exhibit a graph with $\frac{(n-1)(n-2)}{2} + 1$ edges that does *not* have a Hamiltonian cycle.
- (d) Show that a graph on n vertices that has at least $\frac{(n-1)(n-2)}{2} + 2$ edges *has* a Hamiltonian cycle. *Hint: This part of the problem may take more time. Try it last.*

Problem 3

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Problem 4

Let X be a standard Cauchy random variable, *i.e.*, with density $p(x) := \frac{1}{\pi} \cdot \frac{1}{x^2 + 1}$.

(a) Show that $\int_{-\infty}^{+\infty} p(x) dx = 1$.

(b) Compute the probability density function of the random variable $Y := 1/X$.

(c) Consider the random variable X_α with density

$$f_\alpha(x) = \begin{cases} \frac{c_\alpha}{1+x^\alpha}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

where c is a normalizing factor. For what values of α does X_α have a (finite) mean? For what values of α does X_α have a (finite) variance?

Problem 4

Problem 4

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Problem 5

Suppose 52 trees are arranged in a circle and 15 chipmunks live in the trees.

- (a) If we pick a tree at a random position t , compute the expected number of chipmunks in the seven trees $t, t + 1, \dots, t + 6 \pmod{52}$ (a *cyclic run* at t).
- (b) Show, by the probabilistic method, that there is *some* cyclic run of seven trees that contains at least 3 chipmunks.

Problem 5

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