AIM Preliminary Exam: Probability & Discrete Mathematics

September 5, 2010

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let $\mathcal{R}$ be the set of real numbers in the interval $(0, 1)$. Let $\mathcal{S}$ be the set of all *countable* subsets of $\mathcal{R}$. Prove that $\mathcal{R}$ and $\mathcal{S}$ have the same cardinality in the following steps:

(a) Construct an injection from $\mathcal{R}$ to $\mathcal{S}$.

(b) Construct an injection from $\mathcal{S}$ to $\mathcal{R}$. Here you have to use the fact that each element of $\mathcal{S}$ is a countable subset of $\mathcal{R}$.

(c) Draw a conclusion using the Schroeder-Bernstein theorem: the sets $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality if there is an injection from $\mathcal{A}$ to $\mathcal{B}$ and vice versa.

Problem 1

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Problem 2

Let $D(n)$ be the number of permutations of the numbers $1, 2, \ldots, n$ such that i is not in the i-th position for $i = 1, 2, \ldots, n$. Here D stands for "derangement". For example, $(3, 1, 2)$ is a derangement but not $(3, 2, 1)$ because 2 is in the 2-nd position.

(a) Argue that the number of permutations of $1, 2, \ldots, n$ that have any given k numbers in their "original" positions is $(n-k)!$.

(b) Use the inclusion-exclusion principle to justify the following calculation:

$$\begin{aligned} D(n) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\ &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right). \end{aligned}$$

Your solution must state the inclusion-exclusion principle and explain how it applies.

(c) Prove that

$$D(n) = (n-1)\left(D(n-1) + D(n-2)\right), \quad n > 2$$

using the formula in part (b).

(d) Prove the recurrence relation in part (c) by a direct combinatorial argument.

Problem 2

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Problem 3

Three fair dice—each with six faces numbered 1, 2, 3, 4, 5, and 6—are rolled. If no two show the same face, what is the probability that one of the faces is 1?

Problem 3

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Problem 4

Let the probability that a family has exactly n children be $p_n = \alpha r^n$ for $n = 0, 1, 2, \ldots$. Here $\alpha > 0$ and $r \in (0, 1)$ are parameters. Assume that each child is equally likely to be a boy or a girl.

(a) Are the parameters α and r independent? If not, how are they related?

(b) Show that the probability b_k that a family has exactly k boys is given by

$$b_k = \alpha \sum_{n=0}^{\infty} \binom{k+n}{k} \left(\frac{r}{2}\right)^{n+k}, \quad k \geq 1.$$

(c) Evaluate the sum to show that b_k is equal to

$$b_k = \frac{2\alpha r^k}{(2-r)^{k+1}}.$$

Problem 4

Problem 4

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Problem 5

An undirected graph $G = (V, E)$ is *tripartite* if the set of vertices V can be partitioned into three non-empty sets V_1, V_2, and V_3 such that there is no edge from vertex u to v if they belong to the same partition, but there is an edge in E from u to v if they belong to different partitions. Give efficient algorithms to determine if a graph is tripartite in the following two cases:

(a) Give an efficient algorithm to determine if a graph $G = (V, E)$ is tripartite if the graph is given as an adjacency list.

(b) Repeat part (a) if instead the graph is given as an adjacency matrix.

Problem 5

Problem 5

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