AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 5, 2015

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) Let a and b be elements of a Euclidean vector space such that $\|a\|_2 = 3$, $\|a - b\|_2 = 4$, and $\|a + b\|_2 = 6$. Find $\|b\|_2$.

(b) Let A_n be the n-by-n matrix with zeros on the main diagonal and ones elsewhere. That is,

$$A_1 = 0,\ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ldots.$$

Find the determinant of A_n. Check that your answer works for $n = 2$ and 3.

Problem 1

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Problem 2

Given $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, show that
A^*A is an invertible (or nonsingular) matrix if and only if A has full rank.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Find a solution to the differential equation $\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}$.

(b) Show that the origin is a stable fixed point for the system

$$\begin{aligned} x' &= -x^3 + 3x^2y - 3xy^2 + y^3 \\ y' &= x^3 - 3x^2y + 3xy^2 - y^3 \end{aligned}$$

Problem 3

Problem 3

Problem 3

Problem 4

(a) The Laplace transform of $f(t)$ is defined as $F(s) \equiv \int_0^\infty e^{-st} f(t) dt$.

Give a function whose Laplace transform decays more slowly than $1/(s-a)$ as $s \to \infty$, for any real a. The Dirac delta "function" is not allowed. Hint: consider powers of t.

(b) Find a solution to Laplace's equation $\partial_{xx} u + \partial_{yy} u = 0$ in the lower half-plane $(-\infty < x < \infty, -\infty < y \leq 0)$, subject to the boundary condition $u(x, 0) = \sin^2(x)$. Your solution should be bounded by a single constant everywhere in the lower half-plane.

Problem 4

Problem 4

Problem 4

Problem 5

Consider the heat equation with a source term, $\partial_t u = \partial_{xx} u + 1$ on the domain $0 < x < 1, t > 0$.

(a) Let the boundary conditions be $u(0,t) = 0$ and $\partial_x u(1,t) = 0$.

Let the initial condition be $u(x,0) = \sin(\pi x/2)$.

In the limit $t \to \infty$, what is $u(x,t)$?

(b) Now let the equation, the initial condition, and the boundary condition at $x = 1$ be the same, but change the boundary condition at $x = 0$ to $u(0,t) = \sin(t)$. Now in the limit $t \to \infty$, what is $u(x,t)$?

Hint: Try solutions which include e^{it} or $\sin t$ and $\cos t$.

Problem 5

Problem 5

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